

## A NEW CHARACTERIZATION OF THE RADON-NIKODYM PROPERTY

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ABSTRACT. We give geometrical properties of lower semi-continuous convex functions and we obtain a characterization of the Radon-Nikodym property on a Banach space  $X$  using geometrical properties of convex functions defined on  $X$ .

Let  $X$  be a Banach space and  $X'$  its dual. The duality between  $X$  and  $X'$  is denoted by  $\langle \cdot, \cdot \rangle$ . For any function  $f : X \rightarrow ]-\infty, +\infty]$  and any scalar  $\alpha \in ]-\infty, +\infty]$  we define the sets

$$epi f = \{(x, \lambda) \in X \times \mathbb{R}, f(x) \leq \lambda\}, \quad dom f = \{x \in X, f(x) < +\infty\},$$

and

$$S(f, \alpha) = \{(x, \lambda) \in epi f, \lambda \leq \alpha\}.$$

The conjugate function of  $f$  is  $f^*(x') = \sup_{x \in X} \{\langle x', x \rangle - f(x)\}$ . If  $C$  is a non empty subset of  $X$ , we define its indicator function by  $\delta_C(x) = 0$  if  $x \in C$  and  $\delta_C(x) = +\infty$  otherwise. The conjugate function of  $\delta_C$  is  $\delta_C^*(x') = \sup\{\langle x', x \rangle, x \in C\}$ , and the subdifferential of a function  $f$  at a point  $x_0$  is given by

$$\begin{aligned} \partial f(x_0) &= \{x' \in X', \langle x', x_0 \rangle - f(x_0) \geq \langle x', x \rangle - f(x), \forall x \in X\} \\ &= \{x' \in X', f^*(x') = \langle x', x_0 \rangle - f(x_0)\}. \end{aligned}$$

A function  $f$  is rotund at  $x_0 \in X$  if there exist  $x' \in X'$  which verify the following,  $\forall \varepsilon > 0, \exists r > 0$ , such that  $\|x - x_0\| \leq \varepsilon$  whenever

$$\langle x', x_0 \rangle - f(x_0) < \langle x', x \rangle - f(x) + r.$$

A point  $x_0 \in X$  is said to be a strong minimum of  $f$  if the following assertions are fulfill

- i)  $f(x_0) \leq f(x), \forall x \in X$ ,

ii)  $\|x_n - x_0\| \rightarrow 0$  whenever  $f(x_n) \rightarrow f(x_0)$ .

Let  $A \subset X$  a convex closed non empty subset,  $x' \in X'$  and  $\alpha > 0$ , the set

$$S(A, x', \alpha) = \{x \in A, \langle x', x \rangle \geq \delta_A^*(x') - \alpha\}$$

is the slice of  $A$  defined by  $x'$  and  $\alpha$ .

We say that  $x_0 \in A$  is a denting point of  $A$  provided that  $A$  admits slices of arbitrarily small diameter which contain  $x_0$ .

A point  $x_0 \in A$  is said to be strongly exposed provided that there exists  $x' \in X', x' \neq 0$  such that  $x_0 \in S(A, x', \alpha)$  for every  $\alpha > 0$  and these slices have diameters converging to 0 as  $\alpha$  tends to 0. Equivalently,  $x_0$  is strongly exposed by  $x'$  if,  $\langle x', x_0 \rangle = \delta_A^*(x')$  and  $\|x_n - x_0\| \rightarrow 0$  whenever  $\langle x', x_n \rangle \rightarrow \delta_A^*(x')$

A Banach space  $X$  is said to have the Radon Nickodym property (R.N.P) if every nonempty closed bounded convex subset of  $X$  has a denting point. There are several other characterizations of the (R.N.P) (see Bourgin[3], Phelps[7] and Diestel and Hull[6]). The aim of this paper is to study some properties of convex functions and to give a new characterization of the (R.N.P).

**Proposition 1.** *Let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper lower semi-continuous convex function and  $x_0$  be a point in the interior of  $\text{dom} f$ , the following statements are equivalent.*

- 1) *There exists  $x' \in \partial f(x_0)$  such that  $x_0$  is a strong minimum of the function  $x \mapsto f(x) - \langle x', x \rangle$ .*
- 2) *There exists  $x' \in X'$  such that  $(x_0, f(x_0))$  is strongly exposed by  $(x', -1)$  in  $\text{epi} f$ .*
- 3)  *$f$  is rotund at  $x_0$ .*

**Proof.** Let us check the following implications  $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$ . For the first, let  $x' \in \partial f(x_0)$  such that  $f - x'$  admits a strong minimum in  $x_0$ . Choose  $\varepsilon > 0$ ,  $r \leq \min(\frac{\varepsilon}{2\|x'\|}, \varepsilon)$  and using assertions *ii)* of the above definition, we can find  $\delta > 0$  such that

$$|f(x) - \langle x', x \rangle - f(x_0) + \langle x', x_0 \rangle| \leq \delta \Rightarrow \|x - x_0\| \leq r.$$

If  $\alpha \leq \min(\frac{\varepsilon}{2}, \delta)$ , then the diameter of  $S(\text{epi} f, (x', -1), \alpha)$  is less than  $\varepsilon$ . From another hand, suppose that  $(x_0, f(x_0))$  is strongly exposed

by  $(x', -1)$  in  $epif$ . Let  $\varepsilon > 0$  be fixed and take  $\alpha > 0$  such that the diameter of the slice  $S(epif, (x', -1), \alpha)$  is less than  $\varepsilon$ . Observing that every pair  $(x, f(x))$  such that  $f(x) < \langle x', x \rangle - f(x_0) - \langle x', x_0 \rangle + \alpha$  is in  $S(epif, (x', -1), \alpha)$ , we deduce that  $\|x - x_0\| \leq \varepsilon$ ; which show that  $f$  is rotund at  $x_0$  and check the second implication. For the last one, notice that the element  $x'$  which appears in the definition of rotundity is in  $\partial f(x_0)$ . The second assertion *ii*) follow easily from the rotundity.

Notice that in the study of strongly exposed points we can restrain our self to a particular class of linear applications on  $X \times R$  which expose points. Precisely, we have the following lemma. Recall that  $S(f, \alpha) = \{(x, r) \in epif, f(x) \leq r\}$  is the slice of  $epif$  defined by  $\alpha$  and note that if  $\alpha = +\infty$  then  $S(f, \alpha) = epif$ .

**Lemma 1.** *Let  $f : X \rightarrow ]-\infty, +\infty]$  be a convex function,  $\alpha \in ]-\infty, +\infty]$  and  $x_0 \in \text{dom}f$  such that  $f(x_0) < \alpha$ . The pair  $(x_0, f(x_0))$  is strongly exposed in  $S(f, \alpha)$  if and only if there exists  $x' \in \partial f(x_0)$  such that  $(x', -1)$  expose strongly  $(x_0, f(x_0))$  in  $S(f, \alpha)$ .*

The proof use the next lemma as well as the fact that if  $(y', \lambda) \in X' \times R$  expose strongly a point in  $epif$ , then  $(\frac{y'}{\lambda}, -1)$  expose strongly the same point.

**Lemma 2.** *Under the same hypothesis as in lemma 1, if  $x' \in X'$  is such that*

$$\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - r \text{ for every } (x, r) \in S(f, \alpha) - (x_0, f(x_0)).$$

*Then*

$$\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - r \text{ for every } (x, r) \in epif - (x_0, f(x_0))$$

*and we have  $x' \in \partial f(x_0)$ .*

The following proposition play a key role in the main result. Notice that if  $x_0 \in A \subset B$  is a strongly exposed point in  $B$ , then it is also strongly exposed in  $A$ . the reciprocal result is fails in general but it hold in the context of epigraphs.

**Proposition 2.** *Let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper lower semicontinuous convex function, and  $x_0$  a point in the interior of  $\text{dom}f$ . Then*

$(x_0, f(x_0))$  is strongly exposed in  $epi f$  if and only if for every  $\alpha > f(x_0)$  (or  $\exists \alpha > f(x_0)$ ) such that  $(x_0, f(x_0))$  is strongly exposed in  $S(f, \alpha)$ .

**Proof.** Let  $(x_0, f(x_0))$  be strongly exposed in  $S(f, \alpha)$  then, there exists  $x' \in \partial f(x_0)$  such that  $(x', -1)$  expose strongly  $(x_0, f(x_0))$  in  $S(f, \alpha)$ . Let  $(x_n, r_n)_n$  be a sequence in  $epi f$  such that

$$\lim_n [\langle x', x_n \rangle - r_n] = \langle x', x_0 \rangle - f(x_0).$$

Using the convexity of  $f$ , a sequence  $(\xi_n, t_n)_n$  in  $S(f, \alpha)$  such that

$$\lim_n [\langle x', \xi_n \rangle - t_n] = \langle x', x_0 \rangle - f(x_0)$$

and the sequence  $(y_n, s_n)_n$  where  $y_n = \alpha_n x_n + (1 - \alpha)\xi_n$ ,  $s_n = \alpha_n r_n + (1 - \alpha)t_n$  and  $\alpha_n = \frac{\alpha - t_n}{M - t_n}$ , we check that  $(x_n, r_n)_n \in S(f, \alpha)$  from a given rank. We conclude the desired result by using the second characterization of strongly exposed points.

It is well known that a Banach space is an Asplund space if and only if its topological dual has the Radon-Nickodym property (cf, [3]and[7]). In other terms, the Radon-Nickodym property of the dual of a Banach space can be characterized by a property of differentiability of convex functions defined on the space. The main result of this paper show that the (R.N.P) can be characterized directly by a regularity property of convex functions defined on the space and not on its dual. This result can be viewed in the light of corollary 3.2 of [9].

**Theorem 1.** *A Banach space  $X$  has the (R.N.P) if and only if every proper lower semi-continuous bounded below convex function on  $X$  is rotund at last at one point.*

**Proof.** Let  $C$  be a bounded closed convex subset of  $X$ . Its indicator function  $\delta_C$  is proper lower semicontinuous bounded below and convex. So, it admits a point of rotundity  $x_0 \in C$ . then, for every  $\varepsilon > 0$ , there exists  $x' \in X'$  and a real  $r > 0$  such that

$$\langle x', x_0 \rangle < \langle x', x \rangle + r \quad \Rightarrow \quad \|x - x_0\| \leq \varepsilon.$$

Observe that  $\langle x', x_0 \rangle = \delta_C^*(x')$ . We deduce that the slice  $S(C, x', r)$  contains  $x_0$  and has a diameter less than  $\varepsilon$ . So  $x_0$  is a denting point of  $C$ . The necessary part use previous results. Let  $f : X \rightarrow ]-\infty, +\infty]$  be a proper lower semicontinuous bounded below convex function, and

$\alpha > \inf\{f(x), x \in X\}$ . we check that the subset  $\{(x, \lambda), f(x) \leq \lambda \leq \alpha\}$  admits strongly exposed point  $(x_0, f(x_0))$  with  $f(x_0) < \alpha$ , and we conclude by virtue of proposition 1 and proposition 2. It follows from the hypothesis on  $f$  that the slice  $S(f, \alpha)$  is a bounded closed convex non empty subset of  $X \times R$ . Since  $X \times R$  has the (R.N.P), the set  $s - \exp[S(f, \alpha)]$  of all strongly exposed point of  $S(f, \alpha)$  is non empty, and  $S(f, \alpha) = \overline{\text{co}}[s - \exp(S(f, \alpha))]$ , (Bourgin[3], Phelps[7]) where  $\overline{\text{co}}$  stand up for the closed convex hull. If all elements  $(x, f(x))$  of  $s - \exp[S(f, \alpha)]$  are such that  $f(x) = \alpha$  we will have

$$S(f, \alpha) \subset \overline{\text{co}}[(x, \alpha), f(x) = \alpha] = \{(x, \alpha), f(x) \leq \alpha\}$$

which is a contradiction since from the choice of  $\alpha$ , there exists  $x \in X$  such that  $f(x) < \alpha$ .

As a direct consequence of the previous theorem and proposition (1) we get the following result which may be regarded as a variant of a result due to Stegall([8] theorem 5-15).

**Corollary 1.** *A Banach space  $X$  has the (R.N.P) if and only if for every lower semicontinuous bounded below convex function  $f : X \rightarrow ] - \infty, +\infty]$ , there exists  $x' \in X'$  such that  $f - x'$  admits a strong minimum.*

## REFERENCES

- [1] **E. Asplund, 1968**, Frechet differentiability of convex functions, Acta.Math. (121).
- [2] **J. Bourgain, 1979**, La propriété de Radon Nikodym, Pub. Math. de l'université Pierre et Marie curie. N 36.
- [3] **R. D Bourgin, 1983**, Geometric aspects of convex sets with the Radon Nikodym property. Lect Notes in Math., N 993. Springer Verlag.
- [4] **J. B Collier, 1976**, The dual of a space with the Radon Nikodym Property, pacific J. Math. (64), 103-106.
- [5] **J. Diestel, 1985**, Geometry of Banach space, selected topics, Lect. Notes in Math. N 485, Springer Verlag.
- [6] **J. Diestel, J. J Hull, 1977**, Vector measures, Math.Surveys (15), A.M.S.
- [7] **R. Phelps, 1974**, Dentability and extreme points in Banach spaces J. Functional Analysis, (17), 78-90.
- [8] **R. Phelps, 1988**, Convex function, monotone operator and differentiability, Lect. Notes in Math., N 1364, 82-83.

- [9] W.Schachermayer, A.Sersouri, E.Werner, Moduli of non-dentability and the Radon-Nikodym property in Banach spaces. *Israel Journal of Math.*, vol 65, 3 (1989) 225-256.

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