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A Random Multivalued Uniform Boundedness Principle

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Abstract. We present a generalization of the Uniform Boundedness Principle valid for random multivalued linear operators, i.e., multivalued linear operators taking values in the space $L_0(\Omega, Y)$ of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in the Banach space Y. Namely, for a family of such operators that are continuous with positive probability, if the family is pointwise bounded with probability at least $\delta > 0$, then the operators are uniformly bounded with a probability that in each case can be estimated in terms of δ and the index of continuity of the operator. To achieve this result, we develop the fundamental theory of multivalued linear operators on general topological vector spaces. In particular, we exhibit versions of the Closed Graph Theorem, the Open Mapping Theorem, and the Uniform Boundedness Principle for multivalued operators between F-spaces.

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1. Introduction

The theory of random single-valued linear operators began in the 1950's with the work of Itô on operators associated with Markov processes and continued throughout the 1960's with work related to stochastic differential and integral equations; see [3, 9], and [24]. Nowadays, the focus is on operators arising from the modern theory of stochastic integration and stochastic differential equations [20], with applications to such areas as mathematical finance and stochastic control.

On the other hand, the study of multivalued linear operators first appeared in the work of von Neumann [16] on adjoints of linear differential operators. Its prominence today stems primarily from its importance in the field of nonsmooth analysis [8]. A modern development of the theory for the type of multivalued linear operators considered here, but in the context of Banach spaces, appears in the book of Cross [5].

The goal of the present work is to begin the process of merging these two areas of research. Our main result is a random multivalued Uniform Boundedness Principle which generalizes the single-valued version appearing in [25].

Various authors [1, 5, 8, 14, 15] have introduced notions of linearity for multivalued mappings between linear spaces, Banach spaces, or locally convex topological vector spaces. Because the space $L_0(\Omega, Y)$ is an F-space that is neither locally convex nor locally bounded (see below), we must deal with mappings between arbitrary topological vector spaces. We have chosen as our definition of a multivalued linear operator the one which leads to the simplest and most transparent arguments. Namely, we shall assume that our operators have closed affine values, rather than allowing for convex cones as values, as espoused by others.

In Section 2 we present the fundamental principles of linear functional analysis for multivalued linear operators. Versions of the Closed Graph Theorem, Open Mapping Theorem, and Uniform Boundedness Principle (Theorems 2.6, 2.7, and 2.10, respectively) are developed. We refer the reader to [22] for classical terminology and the theory of single-valued maps.

Section 3 is devoted to the study of multivalued linear operators T defined on a Banach space and taking values in the randomization $L_0(\Omega, Y)$ of the Banach space Y. Such an operator T is called a random multivalued linear operator, and those which are continuous are called stochastically continuous, or equivalently, stochastically bounded, in concert with the usual terminology for single-valued operators. The focus here is on the Uniform Boundedness Principle. First we describe the precise manifestation of the Uniform Boundedness Principle developed in Section 2, in the setting of random multivalued linear operators, given the nature of the topology on $L_0(\Omega, Y)$ (Theorem 3.5). We then proceed to prove our main result, Theorem 3.11, called the Random Multivalued Uniform Boundedness Principle. Namely, we introduce the notion of a random multivalued linear operator T that is continuous with positive probability and the corresponding index of continuity $\alpha(T)$ of such an operator. In particular, $\alpha(T) = 1$ means that T is stochastically continuous. Consider a family $\{T_i\}_{i\in I}$ of random multivalued linear operators, each continuous with positive probability, that is pointwise bounded with probability $\delta > 0$, in a sense defined in Section 3. Then the theorem asserts that the operators in the family are stochastically uniformly bounded with positive probability, provided that δ and the $\alpha(T_i)$, $i \in I$, are sufficiently large. More precisely, each T_i is a stochastically bounded multivalued linear operator, with a bound that is independent of $i \in I$, when conditioned on a measurable set whose probability can be estimated in terms of δ and $\alpha(T_i)$. Thus we extend the main result of [25] to the case of random multivalued linear operators.

2. Multivalued Linear Operators

2.1. MULTIVALUED LINEAR OPERATORS

Let X and Y be topological vector spaces. A *multivalued map* will mean a mapping $T: X \to Y$ such that $T(x)$ is a nonempty closed subset of Y for every x in X. A multivalued map $T: X \to Y$ is said to be a *multivalued linear operator* if the two following properties are satisfied:

(i) $T(\lambda x) = \lambda T(x)$ for every scalar $\lambda \neq 0$ and every $x \in X$.
(ii) $T(x) + T(y) \subseteq T(x + y)$, for every $x, y \in X$.

By convention, we set $0T(x) = T(0)$ for all $x \in X$.

The definition of a multivalued linear operator above coincides with the one in [5]. In [14] and [15], in which the setting is purely algebraic, multivalued linear operators are called *linear systems*, while in [1] they are termed *closed linear processes.* The authors of both [5] and [1], however, assume that X and Y are Banach spaces.

Remark 2.1. (1) The assumption that each $T(x)$ be closed is a weak one which allows for the application of familiar analytical methods. Note that if we drop the assumption that each $T(x)$ be closed and we replace (ii) above with $T(x)+T(y) \subseteq \overline{T(x+y)}$, it follows that $\overline{T(x)} + \overline{T(y)} \subseteq \overline{T(x+y)}$, producing a mapping with closed values.

(2) A mapping $T: X \to Y$ is a multivalued linear operator precisely when $T(\lambda x) = \lambda T(x)$ and $T(x) + T(y) = T(x + y)$ for every scalar $\lambda \neq 0$ and $x, y \in X$. Indeed the sufficiency is obvious. For the necessity, suppose that the above assertions hold. Then since

$$T(x + y) + T(-y) \subseteq T(x)$$

for every x, y in X, it follows that

$$T(x + y) \subseteq T(x) - T(-y) = T(x) + T(y),$$

so equality holds. In particular, each multivalued linear operator is an *odd fan* in the sense of [8].

(3) The map $T: X \to Y$ is linear if and only if its graph, namely

$$\text{Graph}\, T = \{(x, y) \in X \times Y : y \in T(x)\},$$

is a linear subspace of $X \times Y$. Indeed, let T be linear, (x_1, y_1) and (x_2, y_2) be in Graph T, and α, β be scalars. Then

$$\alpha y_1 + \beta y_2 \in \alpha T(x_1) + \beta T(x_2) = T(\alpha x_1 + \beta x_2),$$

so $\alpha(x_1, y_1) + \beta(x_2, y_2) \in \text{Graph}\, T$. Conversely, if Graph T is a linear subspace, $y \in T(x)$, and λ is a scalar, then $\lambda(x, y) \in \text{Graph}\, T$ so $\lambda y \in T(\lambda x)$, and it follows that $\lambda T(x) \subseteq T(\lambda x)$. The same inclusion applied to $x' = \lambda x$ and $\lambda' = 1/\lambda$ give the assertion (i) above. Moreover, if $y_1 \in T(x_1)$ and $y_2 \in T(x_2)$, then $(x_1 + x_2, y_1 + y_2) \in \text{Graph}\, T$; that is $y_1 + y_2 \in T(x_1 + x_2)$ which shows (ii).

(4) Note also that $T(0)$ is a (closed) linear subspace of Y, since $\lambda T(0) = T(0)$ and $T(0) + T(0) \subseteq T(0)$.

The following result characterizes multivalued linear operators. Namely, each such multivalued linear operator $T: X \to Y$ corresponds to a closed subspace M of Y and a single-valued linear operator $\tilde{T}: X \to Y/M$.

PROPOSITION 2.2. *Let $T: X \to Y$ be a multivalued linear operator. Then for every $y \in T(x)$ we have $T(x) = y + T(0)$. Consequently if $\pi: Y \to Y/T(0)$ is the quotient map, then there is a single-valued linear operator $\tilde{T}: X \to Y/T(0)$ such that*

$$T(x) = \pi^{-1}(\tilde{T}(x)), \quad x \in X.$$

Proof. Let $y, y' \in T(x)$. Then $y + T(0) \subseteq T(x) + T(0) = T(x)$ and $y' - y \in T(x) - T(x) = T(0)$. Thus $T(x) = y + T(0)$, and the remaining assertions follow. □

In the sequel, if T is a multivalued linear operator, then $\tilde{T}$ will denote the operator defined in Proposition 2.2.

EXAMPLES. (1) Of course, any single-valued linear operator can be regarded as a multivalued linear operator. These are precisely those multivalued linear operators whose values are singletons or, equivalently, those T for which some $T(x)$ is bounded in Y.

(2) As pointed out by R. Cross [5], the simplest examples of multivalued linear operators are obtained as follows. Let $S: Y \to X$ be a continuous, surjective linear operator that is not injective, and let $T(x) = S^{-1}(x)$, $x \in X$.

(3) Let T and T' be two multivalued linear operators from X to Y and λ be a scalar. Then the maps λT and $T + T'$, given by $(\lambda T)(x) = \lambda T(x)$ and $(T + T')(x) = \overline{T(x) + T'(x)}$, respectively, are multivalued linear operators. The Cartesian product of two multivalued linear operators, defined by $(T \times T')(x) = T(x) \times T'(x)$, is a multivalued linear operator from X to $Y \times Y$.

2.2. CONTINUITY OF A MULTIVALUED LINEAR OPERATOR

Unless otherwise indicated, X and Y will continue to be any topological vector spaces.

DEFINITION 2.3. A multivalued linear operator $T: X \to Y$ is called *continuous* if for every neighborhood V of 0 in Y there exists a neighborhood U of 0 in X

such that $T(x) \subseteq T(0) + V$ for every $x \in U$. Choosing V symmetric, it follows immediately from Proposition 2.2 that for such a T and any $x_0 \in X$,

$$T(x) \subseteq T(x_0) + V \quad \text{and} \quad T(x_0) \subseteq T(x) + V$$

for every $x \in x_0 + U$. In fact, T is continuous if and only if the single-valued operator $\tilde{T}$ is continuous with respect to the quotient topology on $Y/T(0)$.

Remark 2.4. Various authors dealing with multivalued mappings introduced notions of continuity and upper and lower semicontinuity (cf. [1, 4, 5, 8, 23]). Our definition of continuity agrees with those appearing in all of these earlier articles when one is dealing with multivalued linear operators, with one exception. Namely, for a multivalued mapping T to be upper semicontinuous at x_0, Castaing and Valadier [4] require the very strong condition (for noncompact-valued mappings) that for every open set V containing $T(x_0)$, there is a neighborhood U of x_0 such that $T(x) \subseteq V$ for all $x \in U$.

Recall that if A and B are nonempty closed subsets of a metric space (E, d), then the *Hausdorff distance* between A and B is defined by

$$\Delta(A, B) = \Delta_E(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(a, B) = d_E(a, B) = \inf_{b\in B} d(a, b)$ (in general this value can be infinite).

Suppose now that X and Y are *F-spaces*, i.e., complete metric topological vector spaces. As is well known, the metrics on such spaces may be assumed to be translation-invariant, an assumption that we shall always make. That is, we assume, for instance, that the metric d_Y on Y satisfies $d_Y(y, y') = d_Y(y + w, y' + w)$ for all $y, y', w \in Y$. If T is a multivalued linear operator from X to Y, then the quotient space $Y/T(0)$ with its usual quotient topology is again an F-space, under the metric

$$\begin{aligned} d_{Y/T(0)}(y + T(0), y' + T(0)) &= \Delta_Y(y + T(0), y' + T(0)) \\ &= \Delta_Y(y - y' + T(0), T(0)) \\ &= d_Y(0, y - y' + T(0)). \end{aligned}$$

(Recall that the translation invariance of d_Y implies that

$$d_Y(y, y' + T(0)) = d_Y(z, y' + T(0))$$

if $y - z \in T(0)$, so $d_{Y/T(0)}$ is well defined and finite-valued.) The following result is now immediate from Proposition 2.2.

PROPOSITION 2.5. *Let X and Y be F-spaces and $T: X \to Y$ be a multivalued linear operator. Then the following assertions are equivalent:*

(i) *T is continuous.*

(ii) $\tilde{T}$ *is continuous.*
(iii) $\Delta(T(x_n),\ T(x)) \to 0$ *whenever* $x_n \to x$ *in* X.

Let us now develop versions of the classical continuity theorems for linear operators based on the Baire Category Theorem, in the context of multivalued linear operators. Several authors have formulated versions of these theorems for multivalued maps in different contexts. See, for example, [1] and [5] for discussions of convex 'processes' and multivalued linear operators, respectively, both in the context of Banach spaces, and [8], dealing with 'fans' in the context of locally convex topological vector spaces.

There are several other extensions of the Uniform Boundedness Principle to multivalued mappings in the literature. For maps satisfying some convexity assumptions and taking values in normed or locally convex spaces, versions of this theorem may be found in [17, 18], and [19]. A uniform boundedness principle for maps between Menger probabilistically normed spaces appears in [7], while a special case of that result valid for random normed spaces was proved in [13]. A version for ordered cones which need not be embeddable in vector spaces appears in [21].

THEOREM 2.6 (Multivalued Closed Graph Theorem). *Let X and Y be F-spaces and $T\colon X \to Y$ be a multivalued linear operator. Then T is continuous if and only if* Graph T *is closed in* $X \times Y$.

Proof. We have

$$\text{Graph}\,T = \{(x, y) : y \in T(x)\} = \{(x, y) : T(x) = y + T(0)\}$$

and

$$\begin{aligned}\text{Graph}\,\tilde{T} &= \{(x, y + T(0)) : y + T(0) = \tilde{T}(x)\}\\ &\subseteq X \times (Y/T(0)) = (X \times Y)/(\{0\} \times T(0)).\end{aligned}$$

Let $\pi\colon X \times Y \to X \times (Y/T(0))$ be the quotient map, given by

$$\pi(x, y) = (x, y + T(0)).$$

Then

$$\text{Graph}\,T = \pi^{-1}(\text{Graph}\,\tilde{T}). \qquad (*)$$

If T is continuous, then $\tilde{T}$ is continuous by Proposition 2.5, so Graph $\tilde{T}$ is closed; hence Graph T is closed, since π is continuous. Conversely, suppose that Graph T is closed. Note that $(*)$ implies that

$$\pi([\text{Graph}\,T]^{c}) = [\text{Graph}\,\tilde{T}]^{c},$$

where A^{c} denotes the complement of a set A. Since π is an open mapping, we see that Graph $\tilde{T}$ is closed. By the classical Closed Graph Theorem, $\tilde{T}$ and hence T are continuous, again by Proposition 2.5. □

We say that the multivalued linear operator T maps X *onto* Y if

$$\bigcup_{x\in X} T(x) = Y.$$

THEOREM 2.7 (Multivalued Open Mapping Theorem). *Let X and Y be F-spaces and $T: X \to Y$ be a continuous multivalued linear operator from X onto Y. Then T is an open mapping, in the sense that for every open set U in X, $T(U) = \bigcup_{x\in U} T(x)$ is open in Y.*

Proof. Let U be open in X and $\pi: Y \to Y/T(0)$ be the quotient map, and recall that $\pi T = \tilde{T}$. By hypothesis, $\tilde{T}$ maps X onto $Y/T(0)$, so by the classical Open Mapping Theorem, $\tilde{T}(U)$ is open in $Y/T(0)$. Hence $T(U) = \pi^{-1}(\tilde{T}(U))$ is open in Y. □

Recall that a subset B of a topological vector space is called *bounded* if for every neighborhood V of 0 there is a number $r > 0$ such that $B \subseteq sV$ for all $s \geqslant r$. A topological vector space is called *locally bounded* if it contains a bounded neighborhood of 0 (such as is the case, for example, for a Banach space).

DEFINITION 2.8. Let $\{T_i\}_{i\in I}$ be a family of continuous multivalued linear operators from X to Y. The collection $\{T_i\}_{i\in I}$ will be called *pointwise bounded* if for every neighborhood V of 0 in Y and every $x \in X$ there exists $r > 0$ such that for all $s \geqslant r$,

$$T_i(x) \subseteq T_i(0) + sV, \quad i \in I;$$

i.e., $\tilde{T}_i(x) \in s\pi_i(V),\ i \in I$, where $\pi_i: Y \to Y/T_i(0)$ is the quotient map. We shall say that $\{T_i\}_{i\in I}$ is *strongly pointwise bounded* if for every $x \in X$ there is a bounded set B_x in Y such that

$$T_i(x) \subseteq T_i(0) + B_x, \quad i \in I.$$

We say that $\{T_i\}_{i\in I}$ is *equicontinuous* if for every neighborhood V of 0 in Y there is a neighborhood U of 0 in X such that $T_i(x) \subseteq T_i(0) + V$ for all $x \in U$ and $i \in I$.

PROPOSITION 2.9. *Let $\{T_i\}_{i\in I}$ be a family of equicontinuous multivalued linear operators from X to Y. Then $\{T_i\}_{i\in I}$ is pointwise bounded. If Y is locally bounded, then $\{T_i\}_{i\in I}$ is strongly pointwise bounded.*

Proof. Let U and V be as in the definition of equicontinuity. We may assume that V is balanced. Given $x \in X$, choose $r > 0$ so that $x \in rU$. Then $r^{-1}x \in U$, so

$$T_i(x) \subseteq r(T_i(0) + V) = T_i(0) + rV, \quad i \in I,$$

and the same then holds for all $s > r$. If Y is locally bounded, then the open set V above can be chosen bounded, and consequently rV is bounded, so $\{T_i\}_{i\in I}$ is strongly pointwise bounded. □

THEOREM 2.10 (Multivalued Uniform Boundedness Principle). *Suppose that X is an F-space. Let $\{T_i\}_{i\in I}$ be a family of continuous, pointwise bounded multivalued linear operators from X to Y. Then $\{T_i\}_{i\in I}$ is equicontinuous.*

Proof. Let V be a balanced neighborhood of 0 in Y and W be another balanced neighborhood such that $W+W+W+W \subseteq V$. By hypothesis, for each $x \in X$ there exists an integer n_x such that $T_i(x) \subseteq T_i(0) + n_x W$ and hence $\tilde{T}_i(x) \in n_x\overline{\pi_i(W)}$ for all $i \in I$ (cf. Proposition 2.2). Thus the closed set

$$E = \{x : \tilde{T}_i(x) \in \overline{\pi_i(W)},\ i \in I\}$$

satisfies

$$\bigcup_{n=1}^{\infty} nE = X.$$

So the Baire Category Theorem implies that there is an integer n_0 such that n_0E has nonempty interior. Let $x_0 \in n_0E$ and U be a neighborhood of 0 in X such that $x_0 + U \subseteq n_0E$. Then for all $i \in I$,

$$\tilde{T}_i(x_0) \in n_0\overline{\pi_i(W)} \subseteq n_0\pi_i(W + W),$$

so

$$T_i(x_0) \subseteq T_i(0) + n_0(W + W).$$

Similarly

$$T_i(x_0 + x) \subseteq T_i(0) + n_0(W + W), \quad x \in U.$$

Thus

$$\begin{aligned} T_i(x) &= T_i(x_0 + x) - T_i(x_0) \\ &\subseteq [T_i(0) + n_0(W + W)] - [T_i(0) + n_0(W + W)] \\ &= T_i(0) + n_0(W + W + W + W) \\ &\subseteq T_i(0) + n_0V, \quad x \in U,\ i \in I. \end{aligned}$$

Let $U' = n_0^{-1}U$. If $x \in U'$, then $T_i(x) \subseteq T_i(0) + V$ for all $i \in I$. Hence $\{T_i\}_{i\in I}$ is equicontinuous. □

COROLLARY 2.11 (Multivalued Banach–Steinhaus Theorem). *Suppose that X is an F-space. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous multivalued linear operators from X to Y and T be a multivalued map from X to Y. Suppose further that for each $x \in X$, $T_n(x) \to T(x)$, in the sense that for every neighborhood V of 0 in Y, $T_n(x) \subseteq T(x) + V$ for all n sufficiently large. Then T is a continuous multivalued linear operator.*

Proof. To see that T is linear, let $x, x' \in X$. If V is any symmetric neighborhood of 0 in Y, then for all n sufficiently large,

$$\begin{aligned} T(x) + T(x') &\subseteq T_n(x) + V + T_n(x') + V \\ &= T_n(x + x') + V + V \\ &\subseteq T(x + x') + V + V + V. \end{aligned}$$

Since V was arbitrary and T has closed values, $T(x) + T(x') \subseteq T(x + x')$. Similarly, $T(\lambda x) = \lambda T(x)$.

If $x \in X$ and V is a balanced neighborhood of 0 in Y, choose another neighborhood W of 0 such that $W + W + W \subseteq V$. For all n sufficiently large and some $y \in Y$ and $s > 1$,

$$\begin{aligned} T_n(x) &\subseteq T(x) + W = y + T(0) + W \\ &\subseteq y + T_n(0) + W + W \subseteq sW + T_n(0) + sW + sW \\ &\subseteq T_n(0) + sV. \end{aligned}$$

It follows that the sequence $\{T_n\}$ is pointwise bounded, so by Theorem 2.10 it is equicontinuous. That is, if V is a neighborhood of 0 in Y and W is another neighborhood of 0 with $W + W + W \subseteq V$, then there is a neighborhood U of 0 in X such that $T_n(x) \subseteq T_n(0) + W$ for all $x \in U$ and $n \geqslant 1$. So for all n sufficiently large,

$$\begin{aligned} T(x) &\subseteq T_n(x) + W \subseteq T_n(0) + W + W \\ &\subseteq T(0) + W + W + W \subseteq T(0) + V, \end{aligned}$$

and hence T is continuous. □

2.3. BOUNDEDNESS OF MULTIVALUED LINEAR MAPS

We deal now with the topic of the boundedness of a multivalued linear operator. We introduce the definition of a bounded operator that will be the most natural one for our setting. Recall from Example 1 that unless T is single-valued, $T(x)$ is never bounded.

DEFINITION 2.12. Let X and Y be topological vector spaces and $T: X \to Y$ be a multivalued linear operator. We say that T is *bounded* if for every neighborhood V of 0 in Y and every bounded set B in X there exists $r > 0$ such that $T(B) \subset T(0) + sV$ for all $s \geqslant r$ (meaning that $\tilde{T}(B) \subset s\pi(V),\ s \geqslant r$). Thus T is bounded if and only if $\tilde{T}$ is bounded. If for every bounded set B in X there exists a bounded set B' in Y such that $T(B) \subseteq T(0) + B'$, then T will be called *strongly bounded*. Note that if X and Y are Banach spaces, then T is bounded if and only if there exists $M > 0$ such that $\Delta(T(x), T(0)) \leqslant M$ for every $x \in X$ with $\|x\| \leqslant 1$, where Δ denotes the Hausdorff distance defined in Section 2.2.

Applying Proposition 2.2, one sees immediately the connection between continuity and boundedness of a multivalued linear operator. It is a generalization of the well-known relationship for single-valued linear operators ([22], Theorem 1.32). The Banach space case follows immediately from the identity

$$\Delta(T(x), T(0)) = \|\tilde{T}(x)\|.$$

PROPOSITION 2.13. *Let $T: X \to Y$ be a multivalued linear operator.*

(i) *If T is continuous, then it is bounded.*
(ii) *If T is continuous and Y is locally bounded, then T is strongly bounded.*
(iii) *If X is an F-space and T is bounded, then T is continuous.*
(iv) *If X and Y are Banach spaces, then T is continuous if and only if there exists $M > 0$ such that*

$$\Delta(T(x), T(0)) \leqslant M\|x\|, \quad x \in X,$$

or equivalently,

$$\Delta(T(x), T(y)) \leqslant M\|x - y\|, \quad x, y \in X.$$

(v) *If X and Y are Banach spaces and T is continuous, then*

$$\begin{aligned} \sup_{\|x\|\leqslant 1} d(0, T(x)) &= \sup_{\|x\|\leqslant 1} \Delta(T(x), T(0)) \\ &= \sup_{\|x\|=1} \Delta(T(x), T(0)) \\ &= \inf\{M > 0 : \Delta(T(x), T(0)) \leqslant M\|x\|,\ x \in X\} \\ &= \|\tilde{T}\|. \end{aligned}$$

If X and Y are Banach spaces, we can regard the supremum above as the *norm* of the multivalued linear operator T and write $\|T\| = \|\tilde{T}\|$ (cf. [5], Section II.1). In this case, of course, Y is locally bounded. Consequently, given a family $\{T_i\}_{i\in I}$ of continuous multivalued linear operators from X to Y, it follows from Definition 2.8 that $\{T_i\}_{i\in I}$ is pointwise bounded precisely when for each $x \in X$ there exists $M_x > 0$ such that

$$\Delta(T_i(x), T_i(0)) \leqslant M_x, \quad x \in X,\ i \in I,$$

and equicontinuous if and only if

$$\sup_{i\in I} \|T_i\| < \infty.$$

Thus the Uniform Boundedness Principle above takes on the following form in the context of Banach spaces (cf. [5], Theorem II.3.15).

THEOREM 2.14 (Multivalued Uniform Boundedness Principle). *Let X and Y be Banach spaces and $\{T_i\}_{i\in I}$ be a family of continuous, multivalued linear operators from X to Y. Suppose that for each $x \in X$ there exists $M_x > 0$ such that*

$$\Delta(T_i(x), T_i(0)) \leqslant M_x, \quad i \in I.$$

Then

$$\sup_{i\in I} \|T_i\| < \infty.$$

COROLLARY 2.15. *Let X and Y be Banach spaces and $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous multivalued linear operators from X to Y. Let T be a multivalued map from X to Y, and suppose that*

$$\lim_{n\to\infty} \Delta(T_n(x), T(x)) = 0, \quad x \in X.$$

Then T is a continuous multivalued linear operator.

3. A Random Multivalued Uniform Boundedness Principle

3.1. MULTIVALUED RANDOM LINEAR OPERATORS

Throughout this section we shall restrict ourselves to considering Banach spaces X and Y, and $L_0(\Omega, Y)$ will denote the F-space of all Bochner random variables on a probability space (Ω, Σ, P) with values in Y. To avoid trivialities and measure-theoretic difficulties, we shall assume that our probability space is complete and nonatomic. Recall that the topology on $L_0(\Omega, Y)$ is that of convergence in probability, with a base of neighborhoods of 0 consisting of $\{U_\varepsilon : 0 < \varepsilon < 1\}$, where

$$U_\varepsilon = \{y \in L_0(\Omega, Y) : P[\|y\| \geqslant \varepsilon] < \varepsilon\}.$$

The usual metric on $L_0(\Omega, Y)$ that reflects this topology is given by

$$d(y_1, y_2) = \int_\Omega \frac{\|y_1 - y_2\|}{1 + \|y_1 - y_2\|} \,\mathrm{d}P.$$

One can easily check that for $0 < \varepsilon < 1$ and $y_1, y_2 \in L_0(\Omega, Y)$,

$$d(y_1, y_2) < \frac{\varepsilon^2}{2} \implies P[\|y_1 - y_2\|] \geqslant \varepsilon] < \varepsilon \implies d(y_1, y_2) < 2\varepsilon,$$

so that

$$B(y, \varepsilon^2/2) \subseteq y + U_\varepsilon \subseteq B(y, 2\varepsilon), \quad y \in L_0(\Omega, Y),\ 0 < \varepsilon < 1,$$

where $B(y, \varepsilon)$ denotes, as usual, the open ball of radius ε about y. Note that a subset E is bounded in $L_0(\Omega, Y)$, relative to the topology of convergence in probability, if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $P[\|y\| \geqslant M_\varepsilon] < \varepsilon$ for all $y \in E$. Such sets E are usually called *stochastically bounded*.

DEFINITION 3.1. In accordance with the single-valued case [25, 26], we shall refer to a multivalued linear operator from X to $L_0(\Omega, Y)$ as a *multivalued random linear operator* from X to Y. If such an operator T is continuous in the sense of Section 2, then we say that it is *stochastically continuous* or *continuous in probability*. Thus T is stochastically continuous if and only if for every $x_n \to x$ in X, $\varepsilon > 0$, and $y \in T(x)$, there exist $y_n \in T(x_n)$, $n = 1, 2, \ldots$, such that $y_n \to y$ in probability. Following Proposition 2.5, we may abbreviate this statement by writing $P[\Delta_Y(T(x_n), T(x)) > \varepsilon] \to 0$ for all $\varepsilon > 0$.

The multivalued random linear operator T from X to Y is called *stochastically bounded* if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that for all $x \in X$ with $\|x\| \leqslant 1$,

$$P[\Delta_Y(T(x), T(0)) \geqslant M_\varepsilon] < \varepsilon,$$

in the sense that for every $y \in T(x)$ there exists $y_0 \in T(0)$ with $P[\|y - y_0\| \geqslant M_\varepsilon] < \varepsilon$. Thus T is stochastically bounded if and only if it is bounded in the sense of Definition 2.12.

Since the space Y is fixed throughout this section, we shall drop the subscript and write Δ for Δ_Y.

Remark 3.2. One must be somewhat careful with the notation specified above. In particular, the expression $\Delta(T(x_1), T(x_2))$ there is 'shorthand' and does not mean a specific function on Ω defined a.s. Rather, it represents any one of a family of functions, as indicated there. Indeed, first of all, the set $E(\omega) = \{y(\omega) : y \in T(x)\}$ clearly makes no sense, since one can choose a specific element in each P-equivalence class of $T(x)$ so that $E(\omega)$ is any subset of Y. And secondly, for most natural choices of T, there are sets $A \in \Sigma$ for which $P(A) > 0$ and the restriction to A of every measurable function is in the set of restrictions $\{y|_A : y \in T(x)\}$ a.s.

The following proposition is now an immediate consequence of the homogeneity of T.

PROPOSITION 3.3. *Let T be a multivalued random linear operator from X to Y. The following assertions are equivalent.*

(i) *T is stochastically continuous.*
(ii) *T is stochastically bounded.*
(iii) *For every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that*

$$P[\Delta(T(x), T(0)) \geqslant M_\varepsilon\|x\|] < \varepsilon, \quad x \in X.$$

(iv) *For every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that*

$$P[\Delta(T(x), T(y)) \geqslant M_\varepsilon\|x - y\|] < \varepsilon, \quad x, y \in X.$$

3.2. STOCHASTIC MULTIVALUED UNIFORM BOUNDEDNESS PRINCIPLE

Let us recast Theorem 2.10 in the context of stochastically continuous multivalued random linear operators. First we introduce the following definition.

DEFINITION 3.4. Let $\{T_i\}_{i\in I}$ be a family of multivalued random linear operators from X to Y. We say that the family is *stochastically pointwise bounded* if for every $x \in X$ and $\varepsilon > 0$ there exists $M_{x,\varepsilon} > 0$ such that

$$P[\Delta(T_i(x), T_i(0)) \geqslant M_{x,\varepsilon}] < \varepsilon, \quad i \in I.$$

Note that this is nothing but the pointwise boundedness of $\{T_i\}_{i\in I}$, in the sense of Definition 2.8. The family $\{T_i\}_{i\in I}$ is *stochastically uniformly bounded* if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$P[\Delta(T_i(x), T_i(0)) \geqslant M_\varepsilon \|x\|] < \varepsilon, \quad x \in X,\ i \in I.$$

Extending Proposition 3.3 to a family of operators, it is clear that equicontinuity for a family of multivalued random linear operators is equivalent to stochastic uniform boundedness. Thus the multivalued Uniform Boundedness Principle can be formulated in the stochastic case as follows:

THEOREM 3.5 (Stochastic Multivalued Uniform Boundedness Principle). *A stochastically pointwise bounded family of stochastically continuous multivalued random linear operators is stochastically uniformly bounded.*

3.3. RANDOM MULTIVALUED UNIFORM BOUNDEDNESS PRINCIPLE

DEFINITION 3.6. Let T be a multivalued random linear operator and Ω_0 be a measurable subset with $P(\Omega_0) > 0$. Define the operator $T|\Omega_0$ by

$$T|\Omega_0(x) = \overline{T(x)|\Omega_0} = \overline{\{y|\Omega_0 : y \in T(x)\}}.$$

We consider $T|\Omega_0$ as an operator with values on the conditional probability space $L_0(\Omega_0, Y)$, endowed with the usual conditional probability $P(\cdot \mid \Omega_0)$, and we call it a *conditional operator* of T. We say that T is (*stochastically*) *continuous with positive probability* if there exists a measurable set Ω_0 with $P(\Omega_0) > 0$ such that $T|\Omega_0$ is stochastically continuous. If T is continuous with positive probability, set

$$\alpha(T) = \sup\{P(\Omega_0) : T|\Omega_0 \text{ is stochastically continuous}\}.$$

The number $\alpha(T)$ quantifies the degree of continuity of T.

We shall now show that the supremum defining $\alpha(T)$ is attained. We begin with some elementary observations.

LEMMA 3.7. *Let T be a multivalued random linear operator from X to Y.*

(i) *Let $\Omega_0' \subseteq \Omega_0$ be measurable. If $T|\Omega_0$ is stochastically continuous, then so is $T|\Omega_0'$.*

(ii) *Let Ω_1 and Ω_2 be measurable. If $T|\Omega_1$ and $T|\Omega_2$ are stochastically continuous, then so is $T|(\Omega_1 \cup \Omega_2)$.*

(iii) *If $\{\Omega_n\}$ is an increasing sequence of measurable sets with union Ω_0 and $T|\Omega_n$ is stochastically continuous for all $n \geqslant 1$, then $T|\Omega_0$ is stochastically continuous.*

Proof. For $\varepsilon > 0$ and $x \in X$, let

$$A(x, \varepsilon) = \{\omega \in \Omega : \Delta(T(x), T(0))(\omega) \geqslant \varepsilon\}$$

(see Remark 3.2).

(i) Let $\lambda = P(\Omega_0')/P(\Omega_0)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x\| < \delta$, then

$$\begin{aligned}\lambda\varepsilon &> P(A(x,\varepsilon) \mid \Omega_0) \\ &= \lambda P(A(x,\varepsilon) \mid \Omega_0') + (1-\lambda) P(A(x,\varepsilon) \mid \Omega_0 \setminus \Omega_0'),\end{aligned}$$

so $P(A(x,\varepsilon) \mid \Omega_0') < \varepsilon$ whenever $\|x\| < \delta$.

(ii) By (i) we may assume that $\Omega_1 \cap \Omega_2 = \emptyset$. Given $\varepsilon > 0$, choose $\delta > 0$ so that for $\|x\| < \delta$,

$$P(A(x,\varepsilon) \mid \Omega_1) < \varepsilon \quad \text{and} \quad P(A(x,\varepsilon) \mid \Omega_2) < \varepsilon.$$

Then as in (i) we have for such x,

$$P(A(x,\varepsilon) \mid \Omega_1 \cup \Omega_2) \leqslant P(A(x,\varepsilon) \mid \Omega_1) + P(A(x,\varepsilon) \mid \Omega_2) < 2\varepsilon.$$

(iii) Given $\varepsilon > 0$, choose n_0 so that $P(\Omega_0 \setminus \Omega_{n_0}) < \varepsilon$, and suppose that $P(A(x,\varepsilon) \mid \Omega_{n_0}) < \varepsilon$ if $\|x\| < \delta$. Set $\lambda = P(\Omega_{n_0})/P(\Omega_0)$, so $1 - \lambda < \varepsilon$. Then for $\|x\| < \delta$,

$$\begin{aligned}P(A(x,\varepsilon) \mid \Omega_0) &= \lambda P(A(x,\varepsilon) \mid \Omega_{n_0}) + (1-\lambda) P(A(x,\varepsilon) \mid \Omega_0 \setminus \Omega_{n_0}) \\ &< \lambda\varepsilon + \varepsilon < 2\varepsilon.\end{aligned}$$

□

PROPOSITION 3.8. *If T is a multivalued random linear operator that is continuous with positive probability, then there exists a measurable set Ω_0 such that $T|\Omega_0$ is stochastically continuous and*

$$P(\Omega_0) = \alpha(T) = \sup\{P(\tilde{\Omega}) : T|\tilde{\Omega} \text{ is stochastically continuous}\}.$$

Proof. By Lemma 3.7(ii) there is an increasing sequence Ω_n of measurable sets with $T|\Omega_n$ stochastically continuous and $P(\Omega_n) \to \alpha(T)$. Now apply (iii). □

Our goal is to develop a version of the Uniform Boundedness Principle for those multivalued random linear operators that are continuous with positive probability. We need the following technical lemma.

LEMMA 3.9. *Let T be a multivalued random linear operator that is continuous with positive probability and $\{x_n\}$ be a sequence which converges to x_0 in X. Then for every $t > 0$,*

$$\liminf_{n\to\infty} P[\Delta(T(x_n), T(0)) \leqslant t] - P[\Delta(T(x_0), T(0)) \leqslant t] \leqslant 1 - \alpha(T).$$

Proof. The meaning of the conclusion is that for every $y_0 \in T(x_0)$ there exist $y_n \in T(x_n),\ n \geqslant 1$, such that $y_n \to y_0$ and

$$\liminf_{n\to\infty} P[\|y_n\| \leqslant t] - P[\|y_0\| \leqslant t] \leqslant 1 - \alpha(T).$$

Suppose first that $\alpha(T) = 1$, i.e., T is stochastically continuous. Then our assertion is that if $y_n \to y_0$ in probability then for all $t > 0$,

$$\liminf_{n\to\infty} P[\|y_n\| \leqslant t] \leqslant P[\|y_0\| \leqslant t].$$

But this follows easily from the well-known fact that random variables that converge in probability also converge in distribution ([12], Section 10.1.c). Indeed, clearly we have $\|y_n\|$ converging to $\|y_0\|$ in probability, hence also in distribution. Thus for $n \geqslant 0$ and $u \geqslant 0$, let $F_n(u) = P[\|y_n\| \leqslant u]$. Then $F_n(u) \to F_0(u)$ at every point u of continuity of F_0. Now, the set of points of discontinuity of F_0 is countable, and F_0 is right-continuous. So for any $t > 0$ and any $\varepsilon > 0$ choose $u > t$ such that F_0 is continuous at u and $F_0(u) < F_0(t) + \varepsilon$. Then

$$\liminf_{n\to\infty} F_n(t) \leqslant \lim_{n\to\infty} F_n(u) = F_0(u) \leqslant F_0(t) + \varepsilon.$$

Thus

$$\liminf_{n\to\infty} F_n(t) \leqslant F_0(t),$$

as desired.

In general, use Proposition 3.8 to pick Ω_0 such that $P(\Omega_0) = \alpha(T)$ and $T|\Omega_0$ is stochastically continuous. Given $y \in T(x_0)$, there exist $y_n \in T(x_n)$, $n \geqslant 1$, such that $y_n|\Omega_0 \to y|\Omega_0$ in probability. For $n \geqslant 0$ write

$$P[\|y_n\| \leqslant t] = P[\|y_n\| \leqslant t \mid \Omega_0]\alpha(T) + P[\|y_n\| \leqslant t \mid \Omega_0^c](1 - \alpha(T)).$$

Then

$$\begin{aligned} &P[\|y_n\| \leqslant t] - P[\|y_0\| \leqslant t] \\ &\quad = (P[\|y_n\| \leqslant t \mid \Omega_0] - P[\|y_0\| \leqslant t \mid \Omega_0])\alpha(T) + \\ &\qquad + (P[\|y_n\| \leqslant t \mid \Omega_0^c] - P[\|y_0\| \leqslant t \mid \Omega_0^c])(1 - \alpha(T)) \\ &\quad \leqslant (P[\|y_n\| \leqslant t \mid \Omega_0] - P[\|y_0\| \leqslant t \mid \Omega_0])\alpha(T) + 1 - \alpha(T). \end{aligned}$$

Choose integers n_k such that

$$P[\|y_{n_k}\| \leqslant t \mid \Omega_0] \longrightarrow \liminf_{n\to\infty} P[\|y_n\| \leqslant t \mid \Omega_0].$$

Then $\{P[\|y_{n_k}\| \leqslant t]\}_{k=1}^{\infty}$ has a convergent subsequence. Thus from our previous argument,

$$\begin{aligned}
&\liminf_{n\to\infty} P[\|y_n\| \leqslant t] - P[\|y_0\| \leqslant t] \\
&\quad\leqslant \Big(\liminf_{n\to\infty} P[\|y_n\| \leqslant t \mid \Omega_0] - P[\|y_0\| \leqslant t \mid \Omega_0]\Big)\alpha(T) + 1 - \alpha(T) \\
&\quad\leqslant 1 - \alpha(T).
\end{aligned}$$

□

DEFINITION 3.10. Let $\{T_i\}_{i\in I}$ be a family of multivalued random linear operators, and let $0 < \delta \leqslant 1$. We say that this family is *pointwise bounded with probability at least* δ if for every $x \in X$ there exists $M_x > 0$ such that

$$P[\Delta(T_i(x), T_i(0)) \leqslant M_x] \geqslant \delta, \quad i \in I.$$

In particular, this will be the case if there is a set Ω_0 with $P(\Omega_0) \geqslant \delta$ such that the family $\{T_i|\Omega_0\}_{i\in I}$ is stochastically pointwise bounded. The converse is false, however, even for single-valued operators, as shown in Example 4 of [25].

THEOREM 3.11 (Random Multivalued Uniform Boundedness Principle). *Let* $\{T_i\}_{i\in I}$ *be a family of multivalued random linear operators that are continuous with positive probability and pointwise bounded with probability at least* δ*, for some* $\delta > 0$*. Then there exists* $M > 0$ *such that*

$$P[\Delta(T_i(x), T_i(0)) \leqslant M\|x\|] \geqslant (2\delta - 1) - (1 - \alpha(T_i)), \quad x \in X,\ i \in I.$$

Proof. Let

$$E_n = \{x : P[\Delta(T_i(x), T_i(0)) \leqslant n] \geqslant \delta\ \forall i \in I\}.$$

Since $X = \bigcup_{n=1}^{\infty} E_n$, by the Baire Category Theorem $\overline{E_{n_0}}$ has nonempty interior for some n_0. So for some $x_0 \in E_{n_0}$ and $r > 0$, we have that $x + x_0 \in \overline{E_{n_0}}$ if $\|x\| < r$.

Fix $i \in I$. For $x \in E_{n_0} - x_0$, choose $y \in T_i(x)$ and $y_0 \in T_i(x_0)$ such that

$$P[\|y + y_0\| \leqslant n_0] \geqslant \delta \quad \text{and} \quad P[\|y_0\| \leqslant n_0] \geqslant \delta.$$

Then

$$P[\|y\| \leqslant 2n_0] \geqslant P[\|y + y_0\| \leqslant n_0,\ \|y_0\| \leqslant n_0] \geqslant 2\delta - 1,$$

i.e.,

$$P[\Delta(T_i(x), T_i(0)) \leqslant 2n_0] \geqslant 2\delta - 1.$$

For any x with $\|x\| < r$, let $\{x_k\} \subseteq E_{n_0} - x_0$ such that $x_k \to x$. Then Lemma 3.9 says

$$\begin{aligned} &P[\Delta(T_i(x), T_i(0)) \leqslant 2n_0] \\ &\quad \geqslant \liminf_{k\to\infty} P[\Delta(T_i(x_k), T_i(0)) \leqslant 2n_0] - (1 - \alpha(T_i)) \\ &\quad \geqslant (2\delta - 1) - (1 - \alpha(T_i)). \end{aligned}$$

The theorem now follows with $M = 2n_0/r$. □

COROLLARY 3.12. *Let $\{T_i\}_{i\in I}$ be a stochastically pointwise bounded family of multivalued random linear operators that are continuous with positive probability. Then for every $0 < \varepsilon < 1$, there exists a constant $M_\varepsilon > 0$ such that*

$$P[\Delta(T_i(x), T_i(0)) \leqslant M_\varepsilon \|x\|] \geqslant \varepsilon - (1 - \alpha(T_i)), \quad x \in X,\ i \in I.$$

Proof. Let $0 < \varepsilon < 1$ be fixed and let $\delta = (\varepsilon + 1)/2$. Using the fact that $\{T_i\}_{i\in I}$ is stochastically pointwise bounded, we get that for every $x \in X$ there exists $M_x > 0$ such that

$$P[\Delta(T_i(x), T_i(0)) \leqslant M_x \|x\|] \geqslant \delta$$

for all $i \in I$. Thus by Theorem 3.11 there exists a constant M_ε (which depends on δ and therefore ε) such that

$$\begin{aligned} P[\Delta(T_i(x), T_i(0)) \leqslant M_\varepsilon \|x\|] &\geqslant (2\delta - 1) - (1 - \alpha(T_i)) \\ &= \varepsilon - (1 - \alpha(T_i)), \quad x \in X,\ i \in I. \end{aligned}$$ □

COROLLARY 3.13. *Let $\{T_i\}_{i\in I}$ be a family of multivalued random linear operators that are continuous with positive probability. Suppose that there exists a measurable subset Ω_0 with $P(\Omega_0) > 0$ such that the family $\{T_i|\Omega_0\}_{i\in I}$ is stochastically pointwise bounded. Then for every $0 < \delta < P(\Omega_0)$ there exists $M_\delta > 0$ such that*

$$P[\Delta(T_i(x), T_i(0)) \leqslant M_\delta \|x\|] \geqslant \delta - (1 - \alpha(T_i)), \quad x \in X,\ i \in I.$$

Proof. The family $\{T_i|\Omega_0\}_{i\in I}$ satisfies the hypotheses of the previous corollary, so for every $0 < \varepsilon < 1$ there exists a constant $M_\varepsilon > 0$ such that

$$P[\Delta(T_i(x), T_i(0)) \leqslant M_\varepsilon \|x\| \mid \Omega_0] \geqslant \varepsilon - (1 - \alpha(T_i|\Omega_0)), \quad x \in X,\ i \in I.$$

For each i, let Ω_i be chosen by Proposition 3.8 so that $\alpha(T_i) = P(\Omega_i)$ and $T_i|\Omega_i$ is stochastically continuous. By Lemma 3.7(i), $T_i|(\Omega_0 \cap \Omega_i)$ is stochastically continuous. In fact,

$$\begin{aligned} P(\Omega_0)\alpha(T_i|\Omega_0) &= P(\Omega_0)P(\Omega_i \mid \Omega_0) \\ &= P(\Omega_i \cap \Omega_0) \\ &\geqslant P(\Omega_i) + P(\Omega_0) - 1 \\ &= P(\Omega_0) - (1 - \alpha(T_i)), \end{aligned}$$

or equivalently

$$P(\Omega_0)(1-\alpha(T_i|\Omega_0)) \leqslant 1-\alpha(T_i).$$

Thus for $0<\delta<P(\Omega_0)$, setting $\varepsilon=\delta/P(\Omega_0)$ and $M_\delta=M_\varepsilon$, we have

$$\begin{aligned} P[\Delta(T_i(x),T_i(0))\leqslant M_\delta\|x\|] &\geqslant P[\Delta(T_i(x),T_i(0))\leqslant M_\delta\|x\| \mid \Omega_0]P(\Omega_0)\\ &\geqslant P(\Omega_0)(\varepsilon-(1-\alpha(T_i|\Omega_0)))\\ &\geqslant \delta-(1-\alpha(T_i)),\quad x\in X,\ i\in I. \qquad \square \end{aligned}$$

3.4. APPLICATIONS

We shall illustrate the applicability of Theorem 3.11 to the theory of stochastic linear systems by considering its application to the stability of multivalued stochastic dynamical systems. Natural examples of such systems are linear stochastic control systems and linear systems governed by stochastic differential inclusions. We refer the reader to [6] for the properties of Banach-space-valued dynamical systems and to [9] and [10] for studies of stability for stochastic systems. An elementary description of stochastic control systems determined by stochastic differential equations appears in §IX of [20].

Consider a nonanticipating, multivalued, linear stochastic dynamical system S, defined for $t\geqslant 0$, with state space the Banach space X. Stationarity (time-homogeneity) need not be assumed for the present discussion, but it is a common feature of such systems. The outputs of the system are then stochastic processes $[Y_t,\ t\geqslant 0]$, where each $Y_t\in L_0(\Omega,X)$ for some probability space (Ω,Σ,P).

For each $t\geqslant 0$, let Φ_t be the corresponding (multivalued) state transition map, i.e., Φ_t maps $x\in X$ to the output state variables Y_t produced by inputs with initial state x. If $\Phi_t(0)$ is closed in $L_0(\Omega,X)$, then Φ_t is a multivalued random linear operator. Let us assume that $\Phi_t(0)$ is closed for all $t\geqslant 0$. Set

$$Z(S)=\{[Y_t,\ t\geqslant 0]: Y_t\in\Phi_t(0)\ \forall t\geqslant 0\}.$$

The elements of $Z(S)$ will be called *null-initial outputs*.

DEFINITION 3.14. The system S has *bounded outputs modulo null-initial outputs* [*bounded outputs modulo null-initial outputs with probability at least* δ] if the operators $\Phi_t,\ t\geqslant 0$ are stochastically pointwise bounded, in the sense of Definition 3.4 [resp., pointwise bounded with probability at least δ, in the sense of Definition 3.10].

Applying Theorem 3.11 and Corollary 3.12 we obtain the following:

THEOREM 3.15. *Let S be a nonanticipating, multivalued, linear stochastic dynamical system, defined for $t\geqslant 0$, with state space the Banach space X. Assume* *that the state transition maps Φ_t, $t \geqslant 0$, are multivalued random linear operators that are continuous with positive probability. If the system has bounded outputs modulo null-initial outputs with probability at least δ, then there exists $M > 0$ such that*

$$P[\Delta(\Phi_t(x), \Phi_t(0)) \leqslant M\|x\|] \geqslant (2\delta - 1) - (1 - \alpha(\Phi_t)), \quad x \in X,\ t \geqslant 0.$$

In particular, if S has bounded outputs modulo null-initial outputs, then for every $0 < \varepsilon < 1$,

$$P[\Delta(\Phi_t(x), \Phi_t(0)) \leqslant M_\varepsilon\|x\|] \geqslant \varepsilon - (1 - \alpha(\Phi_t)), \quad x \in X,\ t \geqslant 0.$$

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