EXTREMALITY AND DIFFERENTIABILITY OF CONVEX FUNCTIONS

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Abstract. We study relations between the differentiability and some geometrical properties of convex functions defined on a Banach space. As a consequence, we get a characterization of Radon-Nikodym property in Banach spaces by geometrical properties of convex functions.

Introduction.

In this paper, we study some geometrical properties in relation with differentiability of convex functions defined on a Banach space. By geometrical properties, we mean properties of rotundity, introduced by Asplund and Rockafellar in [2] as well as some properties of extremality (existence of extremal, denting, exposed, and strongly exposed points) into the epigraph of convex functions.

First, we examine the relations, in the case of convex functions, between the different geometrical properties. On the other hand we study the duality between those properties and the differentiability. Our approach, which consists in working systematically on the epigraph, differs from those used by Asplund-Rockafellar [2], Bourgain [3], Collier [8] and Phelps [16],[17], whom explored all of those properties, but under other aspects. We propose elementary techniques and we rest on the traditional tools of convex analysis. In the first section, we introduce the definitions of differentiability, rotundity and extremality; and we begin by a characterization of strictly convex functions by a property of extremality. In the second section, we highlight the link between strict convexity (resp. rotundity) of a convex function and the existence of an exposed (resp. strongly exposed) point in its epigraph. We also show that the property of extremality in the epigraph is equivalent to the same extremality property in a "lower section" of the epigraph. This last property is not granted because a subset of the epigraph can admit for example an extremal point even if this point is not an extremal point in the epigraph. In the third section, we first consider the positively homogeneous functions. For this type of functions, we establish a duality result between differentiability and the existence of a strongly exposed point. This result is similar to that given [16] for semi-norms. Then, we show that differentiability of convex functions can be brought back to that of positively homogeneous ones. As a consequence, we give a characterization of the differentiability of a function by a property of extremality in the epigraph of its conjugate function. Finally, we deduce a result of Asplund-Rockafellar ([2], Theorem 1, page 450). In the last section, we give a new

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characterization of Radon-Nikodym property by rotundity and also by some equivalent properties of convex functions.

1. Definitions and Preliminary Results.

Let X be a Banach space, X' its topological dual, B_X its closed unit ball and $B(x,r)$ the open ball centered at x with radius r. We denote by $\langle x',x\rangle$ the $\langle X',X\rangle$ duality, and by $\sigma(X,X')$ the weak topology of X. Let A be a subset of X. We will denote by $int(A)$, (resp. $co(A)$, $\overline{co}(A)$, $\delta^\star(.,A)$) the interior, (resp. the convex hull, the closed convex hull, the support function) of A. Recall that $\delta^\star(x',A) = \sup\{\langle x',x\rangle, x \in A\}$. It is easy to check that $\delta^\star(.,A) = \delta^\star(.,\overline{co}(A))$. For $\alpha > 0$ and $x' \in X'$, $x' \neq 0$, we define the set $S(A,x',\alpha) = \{x \in A, \langle x',x\rangle > \delta^\star(x',A) - \alpha\}$, which is called the slice of A defined by x' and α. Let K be a convex subset of X and $e \in K$. We say that:

- e is an extremal point in K if for every $x,y \in K$, $e = \frac{1}{2}(x+y)$ then $x = y = e$.
- e is an exposed point in K if there exists $y' \in Y'$ such that $\langle y',e\rangle = \delta^\star(y',K)$ and $\langle y',e\rangle > \langle y',x\rangle$ for every $x \in K$, $x \neq e$.
- e is a point of continuity (P.c) in K if the identity mapping $id : (K,\sigma(X,X')) \longrightarrow (K,\|.\|)$ is continuous at e.
- e is a denting point in K if $e \notin \overline{co}(K \setminus B(e,\varepsilon))$, for every $\varepsilon > 0$; or equivalently, if for every $\varepsilon > 0$, there exists a slice $S(K,x',\alpha)$ in K, the diameter of which is less than ε.
- e is a strongly exposed point in K if there exists $y' \in Y'$ such that $\langle y',e\rangle = \delta^\star(y',K)$ and $\|y_n - e\| \to 0$ whenever $(y_n)_n$ is a sequence in K such that $\langle y',y_n\rangle \to \langle y',e\rangle$; or equivalently, if there exists $x' \in X'$ such that the diameter of the slices $S(K,x',\alpha)$ tends to 0 when $\alpha \to 0$.

In the sequel, we shall write $Ext(K)$, (resp. $Exp(K)$, $P.c(K)$, $Dent(K)$, $S\text{-}exp(K)$) to denote the set of all extremal, (resp. exposed, of continuity, denting, strongly exposed) points in K. We can easily check that $S\text{-}exp(K) \subset Exp(K) \cap Dent(K)$ and $Dent(K) \subset Ext(K) \cap P.c(K)$.

For a function $f : X \longrightarrow]-\infty,+\infty]$, the epigraph, domain and graph of f are

$$epi f = \{(x,\lambda) \in X \times \mathbb{R}, \quad f(x) \leq \lambda\}, \qquad dom f = \{x \in X, \quad f(x) < +\infty\}$$

$$\text{and} \qquad Gr(f) = \{(x,f(x)), \quad x \in dom f\}.$$

For every $\alpha \in \mathbb{R}$, put $S(f,\alpha) = \{(x,\lambda) \in epi f, \lambda \leq \alpha\}$ and $\{f \leq \alpha\} = \{x \in X, f(x) \leq \alpha\}$. A function $f : X \to]-\infty,+\infty]$ is convex if $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for every $x,y \in X$, and every $t \in [0,1]$. It is strictly convex at x_0 if $f(x_0) < tf(x)+(1-t)f(y)$ for every $t \in]0,1[$ and $x,y \in X$ such that $x_0 = tx+(1-t)y$. The conjugate or polar of a function $f : X \to]-\infty,+\infty]$ is the function $f^\star$ with values in $]-\infty,+\infty]$ defined for each $x' \in X'$ by $f^\star(x') = \sup\{\langle x',x\rangle - f(x), x \in X\}$. Notice that, when X' is equipped with its norm or the $\sigma(X',X)$ weak topology, f^* is a lower semicontinuous convex function. Let C be a subset of X. Classically, δ_C denotes the indicator function of C, i.e., $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if not. Its polar $\delta^\star_C(x)$ is the support function $\delta^\star(.,C)$ of C, which justifies the last notation. The subdifferential of a function $f : X \to]-\infty, +\infty]$ is the (perhaps empty) set defined for every point x_0 in X by

$$\partial f(x_0) = \{x' \in X', \langle x', x_0\rangle - f(x_0) \geq \langle x', x\rangle - f(x), \quad \forall x \in X\}.$$

A function $f : X \to]-\infty, +\infty]$ is said to be Gateaux differentiable at $x_0 \in X$ if for every $x \in X$ the limit $f'(x_0)(x) = \lim\limits_{t\to 0} \frac{f(x_0+tx)-f(x_0)}{t}$ is finite and $f'(x_0) \in X'$. If this limit is uniform on $x \in B_X$, the function f is said to be Frechet differentiable at x_0. Notice that f is Frechet differentiable at x_0 if there exists $f'(x_0) \in X'$ such that $\lim\limits_{x\to 0} \frac{f(x_0+x)-f(x_0)-f'(x_0)(x)}{\|x\|} = 0$. We say that a function $f : X \to]-\infty, +\infty]$ is rotund at $x_0 \in X$ if there exists $x' \in X'$ which verifies the following property: $\forall \varepsilon > 0, \exists r > 0$,

$$\langle x', x_0\rangle - f(x_0) < \langle x', x\rangle - f(x) + r \quad \Rightarrow \|x - x_0\| \leq \varepsilon; \quad x \in X.$$

Remarks 1.1. *1) Let x' be in X' such that $(x', -1)$ exposes $epi f$ at $(x_0, f(x_0))$. Then, $x' \in \partial f(x_0)$. Indeed, for every $x \in X$, we have*

$$\langle x', x_0\rangle - f(x_0) \geq \langle x', x\rangle - f(x).$$

2) The element x' which appears in the definition of rotundity is in $\partial f(x_0)$. Indeed, if $x \neq x_0$ then $\|x - x_0\| > 0$ and there exists $r > 0$ such that

$$\langle x', x_0\rangle - f(x_0) \geq \langle x', x\rangle - f(x) + r \geq \langle x', x\rangle - f(x).$$

Thus, if f is rotund at x_0, then $\partial f(x_0) \neq \emptyset$ and $\sup\{\langle x', x\rangle - f(x), x \in X\}$ is reached at x_0 and only at this point. Consequently, f is rotund at x_0 if and only if there exists $x' \in X'$ such that $\forall\varepsilon > 0$, $\exists r > 0$, $f^*(x') < \langle x', x\rangle - f(x) + r$ implies $\| x - x_0 \| \leq \varepsilon$.
3) If a convex function f is rotund at x_0, then it is strictly convex at this point. If not, there exists $x_1 \neq x_2$ in X and $\alpha \in]0, 1[$ such that $x_0 = \alpha x_1 + (1 - \alpha)x_2$ and $f(x_0) = f(\alpha x_1 + (1 - \alpha)x_2) = \alpha f(x_1) + (1 - \alpha)f(x_2)$.
4) The following property is due to Castaing-Valadier [7]. We will often use it in the sequel. Let X and Y be two spaces in a separating duality and $f : X \to]-\infty, +\infty]$ a lower semicontinuous convex function. The polar $f^\star$ of f is defined from Y into $]-\infty, +\infty]$. Then, we have $\delta^\star{}_{epi\, f^\star}(x, r) = -rf(\frac{-x}{r})$ for every $r < 0$.
In the particular case where $r = -1$, we have $\delta^\star_{epif^\star}(x, -1) = f(x)$ for every $x \in X$ and $f^\star(y) = \delta^\star_{epif}(y, -1)$ for every $y \in Y$.

Using the previous definitions, it is easy to check the following facts

i) $Ext(epif) \subset Gr(f)$.
ii) $(x_0, f(x_0))$ is exposed in $epif$ if and only if $\partial f(x_0) \neq \emptyset$ and $\partial f(x_0) \cap \partial f(x) = \emptyset$, for every $x \neq x_0$.
iii) $Exp(epif) \subset Ext(epif) \cap \{(x, f(x)), \partial f(x) \neq \emptyset\}$.
iv) Let f be a strictly convex function. Then, for every $x' \in X'$, the function g defined by $g(x) = \langle x', x\rangle - f(x)$ reaches its maximum at only one point.

We can now formulate a first result which characterizes strictly convex functions using some properties of extremality in the epigraph.

Proposition 1.2. *Let $f : X \to]-\infty, +\infty]$ be a convex function. Then, f is strictly convex and $\partial f(x) \neq \emptyset$ for every $x \in domf$ if and only if $Exp(epif) = Ext(epif) = Gr(f)$.*

Proof. For necessity, we have only to show that $Gr(f) \subset Exp(epif)$, the other inclusions are obvious. Let $(x_0, f(x_0)) \in Gr(f)$ and $x' \in \partial f(x_0)$. We have

$$\langle x', x_0 \rangle - f(x_0) \geq \langle x', x \rangle - f(x), \quad \forall x \in X.$$

By virtue of iv) above this inequality is strict and so

$$\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - \lambda, \quad \forall (x, \lambda) \in epif \text{ with } x \neq x_0.$$

In the particular case where $x = x_0$, we get $\lambda \neq f(x_0)$ and consequently $f(x_0) < \lambda$, which implies $\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - \lambda$. Thus, we get

$$\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - \lambda, \quad \forall (x, \lambda) \in epif \text{ and } (x, \lambda) \neq (x_0, f(x_0)).$$

That is $(x_0, f(x_0))$ is an exposed point by $(x', -1)$ in $epif$. For the sufficiency, notice that $\partial f(x) \neq \emptyset$ is obvious since, if $x \in domf$, then $(x, f(x)) \in Gr(f)$ and $(x, f(x)) \in Exp(epif)$ which implies $\partial f(x) \neq \emptyset$.
If f is not strictly convex, then there exist $x_1 \neq x_2$ in $domf$ and $\alpha \in]0, 1[$ such that

$$f(\alpha x_1 + (1 - \alpha)x_2) = \alpha f(x_1) + (1 - \alpha) f(x_2).$$

Putting $a = \alpha x_1 + (1 - \alpha)x_2$, we get $(a, f(a)) = \alpha(x_1, f(x_1)) + (1 - \alpha)(x_2, f(x_2))$, with $(x_1, f(x_1)) \neq (x_2, f(x_2))$. Then, $(a, f(a))$ should not be an extremal point in $epif$; which contradicts the hypothesis and completes the proof.■

2. Extremality in the Epigraph of a Convex Function.

Our goal in this section is to show that the extremal properties in the epigraph can be formulated in terms of some properties in the "slices" of this one. The space $X \times \mathbb{R}$ will be endowed with the following norm $\|(x, r)\| = \max\{\|x\|, |r|\}$.

Lemma 2.1. *Let $f : X \to]-\infty, +\infty]$ be a convex function, $x_0 \in domf$ and let $\alpha \in]-\infty, +\infty]$ such that $f(x_0) < \alpha$. Set $S(f, \alpha) = \{(x, r) \in epif, r \leq \alpha\}$. If $x' \in X'$ verifies $\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - r$ for every $(x, r) \in S(f, \alpha)$ and $(x, r) \neq (x_0, f(x_0))$, then, $\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - r$, for every $(x, r) \neq (x_0, f(x_0)$ in $epif$; and consequently $x' \in \partial f(x_0)$.*

Proof. Notice that if $\alpha = +\infty$, then $S(f, \alpha) = epif$ and the result is obvious in this case. If $\alpha < +\infty$, let $(x, r) \in epif \setminus S(f, \alpha)$ and $t = \frac{\alpha - f(x_0)}{r - f(x_0)}$; then, $t \in]0, 1[$ since $f(x_0) < \alpha < r$. Setting $y = tx + (1 - t)x_0$, we have $f(y) \leq tf(x) + (1 - t)f(x_0) \leq tr + (1 - t)f(x_0) = \alpha$. Thus, $(y, f(y)) \in S(f, \alpha)$ and consequently

$$\begin{aligned} \langle x', x_0 \rangle - f(x_0) &> \langle x', y \rangle - f(y) \\ &\geq t\langle x', x \rangle + (1 - t)\langle x', x_0 \rangle - tr - (1 - t)f(x_0) \\ &= t(\langle x', x \rangle - r) + (1 - t)(\langle x', x_0 \rangle - f(x_0)). \end{aligned}$$

So, $\langle x', x_0 \rangle - f(x_0) > \langle x', x \rangle - r$. The last assertion is obvious.■

Corollary 2.2. *Let $f : X \to]-\infty, +\infty]$ be a convex function, $x_0 \in domf$ and $x' \in X'$. Then, $(x_0, f(x_0))$ is an exposed point by $(x', -1)$ in $epif$ if and only if $(x_0, f(x_0))$ is exposed by $(x', -1)$ in every slice $S(f, \alpha)$ with $\alpha > f(x_0)$.*

Proof. If $(x', -1)$ exposes $epif$ at $(x_0, f(x_0))$, then for every $\alpha > f(x_0)$, it exposes $S(f, \alpha)$ at the same point $(x_0, f(x_0))$. Conversely, if there exists $\alpha > f(x_0)$ such that $(x', -1)$ exposes $S(f, \alpha)$ at $(x_0, f(x_0))$, then by virtue of the previous lemma, the inequality $\langle x', x_0\rangle - f(x_0) > \langle x', x\rangle - r$ holds for every $(x, r) \in epif$. Which means that $(x', -1)$ exposes $epif$ at $(x_0, f(x_0))$.■

Corollary 2.3. *Let $f : X \to]-\infty, +\infty]$ be a convex function, $\alpha \in]-\infty, +\infty]$ and $x_0 \in domf$ such that $f(x_0) < \alpha$. Then, $(x_0, f(x_0))$ is exposed (resp. strongly exposed) in $S(f, \alpha)$ if and only if, there exists $x' \in \partial f(x_0)$ such that $(x', -1)$ exposes (strongly exposes) $(x_0, f(x_0))$ in $S(f, \alpha)$.*

Proof. It is obvious that the condition is sufficient. Conversely, if $(x_0, f(x_0))$ is an exposed point in $S(f, \alpha)$, then there exists $y' \in X'$ and $\lambda \in \mathbb{R}$ such that $\langle y', x_0\rangle + \lambda f(x_0) = \delta^{\star}_{S(f,\alpha)}(y', \lambda) \geq \langle y', y\rangle + \lambda r$ for every (y, r) in $S(f, \alpha)$ with $(y, r) \neq (x_0, f(x_0))$. Choosing (y, r) such that $f(x_0) < r < \alpha$ and $y = x_0$, we get $\langle y', x_0\rangle + \lambda f(x_0) > \langle y', x_0\rangle + \lambda r$ which implies that $\lambda < 0$. If we set $x' = -\frac{y'}{\lambda}$, then we obtain

$$\langle x', x_0\rangle - f(x_0) = \delta^{\star}_{S(f,\alpha)}(x', -1) > \langle x', y\rangle - r,$$

for every (y, r) in $S(f, \alpha)$ such that $(y, r) \neq (x_0, f(x_0))$. That is $(x_0, f(x_0))$ is an exposed point by $(x', -1)$ in $S(f, \alpha)$. Using Lemma 2.1, we get that $x' \in \partial f(x_0)$. Suppose now that $(x_0, f(x_0))$ is a strongly exposed point by (y', λ) in $S(f, \alpha)$ and set $x' = \frac{-y'}{\lambda}$. Let $(x_n, r_n)_n$ be a sequence in $S(f, \alpha)$ such that $\lim_n(\langle x', x_n\rangle - r_n) = \langle x', x_0\rangle - f(x_0)$ that is

$$\lim_n(\langle y', x_n\rangle + \lambda r_n) = \langle y', x_0\rangle + \lambda f(x_0).$$

One obtains $\|(x_n, r_n) - (x_0, f(x_0))\| \to 0$, which show that $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $S(f, \alpha)$ and that $x' \in \partial f(x_0)$.■

We are now ready to examine a first relation between smoothness and geometrical properties of the epigraph.

Proposition 2.4. *Let $f : X \to]-\infty, +\infty]$ be a lower semicontinuous convex function and $x_0 \in domf$. Then,*

1) *f is strictly convex and subdifferentiable at $x_0 \Longleftrightarrow (x_0, f(x_0))$ is exposed in $epif$.*
2) *f is rotund at $x_0 \Longleftrightarrow (x_0, f(x_0))$ is a strongly exposed point in $epif$.*

Proof. Notice that the first assertion is another formulation of Proposition 1.2. Suppose that f is strictly convex and subdifferentiable at x_0. For $x' \in \partial f(x_0)$, we have

$$\langle x', x_0\rangle - f(x_0) \geq \langle x', y\rangle - \lambda, \quad \forall (y, \lambda) \in epif.$$

With respect to the $\langle X \times \mathbb{R}, (X \times \mathbb{R})'\rangle$ duality, the last inequality means that

$$\langle (x', -1), (x_0, f(x_0))\rangle \geq \langle (x', -1), (y, \lambda)\rangle, \quad \forall (y, \lambda) \in epif. \tag{1}$$

Since $(x_0, f(x_0))$ is in $epif$, we get $\langle (x', -1), (x_0, f(x_0)) \rangle = \delta^\star_{epif}(x', -1)$. We must check that the inequality (1) is strict for every (y, λ) in $epif$ with $(y, \lambda) \neq (x_0, f(x_0))$. If $y = x_0$, then $f(x_0) < \lambda$ for $\lambda \neq f(x_0)$ in this case. This implies

$$\langle x', x_0 \rangle - f(x_0) > \langle x', y \rangle - \lambda.$$

That is $\langle (x', -1), (x_0, f(x_0)) \rangle > \langle (x', -1), (y, \lambda) \rangle$. Now suppose $y \neq x_0$ and $\langle x', x_0 \rangle - f(x_0) = \langle x', y \rangle - \lambda$. Since $f(y) \leq \lambda$ and $x' \in \partial f(x_0)$, we have again $\langle x', x_0 \rangle - f(x_0) = \langle x', y \rangle - f(y)$. So,

$$(1-\alpha)\langle x', y \rangle - (1-\alpha)\langle x', x_0 \rangle + (1-\alpha) f(x_0) + \alpha f(x_0) = (1-\alpha) f(y) + \alpha f(x_0)$$

where $\alpha \in]0, 1[$ is arbitrary. But f being strictly convex, we deduce that

$$f(\alpha x_0 + (1-\alpha)y) < \langle x', (\alpha x_0 + (1-\alpha)y) \rangle - \langle x', x_0 \rangle + f(x_0);$$

which contradicts the fact that $x' \in \partial f(x_0)$. Consequently, the inequality (1) is strict and thus $(x_0, f(x_0))$ is exposed by $(x', -1)$ in $epif$. Conversely, if $(x_0, f(x_0))$ is exposed in $epif$, then by virtue of Remark 1.1 we have $\partial f(x_0) \neq \emptyset$. Write $x_0 = tx_1 + (1-t)x_2$, with $t \in]0, 1[$, (x_1, x_2) in $X \times X$ and take $x' \in \partial f(x_0)$ such that $(x', -1)$ exposes $(x_0, f(x_0))$. Since $\langle x', x_i \rangle - f(x_i) < \langle x', x_0 \rangle - f(x_0)$ for $i = 1, 2$, we have $\langle x', x_0 \rangle - t f(x_1) - (1-t) f(x_2) < \langle x', x_0 \rangle - f(x_0)$. So, $f(tx_1 + (1-t)x_2) = f(x_0) < t f(x_1) + (1-t) f(x_2)$; which shows that f is strictly convex at x_0.

For the second equivalence, suppose that f is rotund at x_0. Then, there exists $x' \in X'$ such that $\forall \varepsilon > 0, \exists \alpha > 0$ such that all slices $S(epif, (x', -1), \alpha)$ have their diameter less than ε. Indeed, using the definition of the rotundity, there exists $x' \in \partial f(x_0)$ such that for every fixed $0 < \delta \leq min(\frac{\varepsilon}{2}, \frac{\varepsilon}{4.\|x'\|})$, there exists $r > 0$ such that

$$f(x) \leq \langle x', x \rangle + f(x_0) - \langle x', x_0 \rangle + r \Rightarrow \|x - x_0\| \leq \delta.$$

Consider the slice $S(epif, (x', -1), \alpha)$ with $\alpha = min(r, \frac{\varepsilon}{2})$. We have

$$\begin{aligned} S(epif, (x', -1), \alpha) &= \{(x, \lambda), f(x) \leq \lambda, \quad \langle x', x \rangle - \lambda > \delta^\star_{epif}(x', -1) - \alpha\} \\ &= \{(x, \lambda), f(x) \leq \lambda, \quad \langle x', x \rangle - \lambda > f^\star(x') - \alpha\} \\ &= \{(x, \lambda), f(x) \leq \lambda, \quad \langle x', x \rangle + f(x_0) - \langle x', x_0 \rangle + \alpha > \lambda\}. \end{aligned}$$

Thus if (x_1, λ_1) and (x_2, λ_2) are in $S(epif, (x', -1), \alpha)$ then,

$$\langle x', x_i \rangle + f(x_0) - \langle x', x_0 \rangle + \alpha > \lambda_i \geq f(x_i) \geq \langle x', x_i \rangle + f(x_0) - \langle x', x_0 \rangle \text{ for } i = 1, 2.$$

So, we deduce from one hand that $\|x_1 - x_2\| \leq 2\delta \leq \varepsilon$ by virtue of the rotundity of f at x_0, and from the other hand that $|\lambda_1 - \lambda_2| \leq \|x'\| \|x_1 - x_2\| + \alpha \leq \varepsilon$, which shows that the diameter of $S(epif, (x', -1), \alpha)$ is less than ε. Conversely, suppose that $(x_0, f(x_0))$ is strongly exposed in $epif$. Then, there exists $x' \in \partial f(x_0)$ such that $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $epif$. Let $\varepsilon > 0$. Then, there exists $\alpha > 0$ such that the diameter of $S(epif, (x', -1), \alpha)$ is less than ε. If x verifies

$$f(x) < \langle x', x \rangle + f(x_0) - \langle x', x_0 \rangle + \alpha,$$

then, $\langle x', x\rangle - f(x) > f^\star(x') - \alpha = \delta^\star_{epif}(x', -1) - \alpha$; which shows that $(x, f(x)) \in S(epif, (x', -1), \alpha)$. Thus $\|x - x_0\| \leq \varepsilon$, which completes the proof of the converse.■

From the previous result we can get the following Corollary which characterizes the rotundity of a convex function as in Proposition 1.2 for a strict convex one.

Corollary 2.5. *Let $f : X \to]-\infty, +\infty]$ be a lower semicontinuous convex function. Then, f is rotund at each $x \in domf$ if and only if S-$exp(epif) = Dent(epif) = Exp(epif) = Ext(epif) = Gr(f)$.*

Let $A \subset B$ be two subsets of a Banach space X. It is obvious that if $x_0 \in A$ is an extremal (resp. denting, of continuity, strongly exposed) point in B, then it is an extremal (resp. denting, of continuity, strongly exposed) point in A. Generally, the converse result does not hold. In the particular case of epigraphs and theirs slices, it holds. More precisely, we have the following result,

Proposition 2.6. *Let $f : X \to]-\infty, +\infty]$ be a lower semicontinuous convex proper function. The following assertions hold:*
1) $(x_0, f(x_0)) \in Ext(epif) \iff \forall\alpha > f(x_0)$ (or $\exists\alpha > f(x_0)$), $(x_0, f(x_0)) \in Ext(S(f, \alpha))$.
2) $(x_0, f(x_0)) \in P.c(epif) \iff \forall\alpha > f(x_0)$ (or $\exists\alpha > f(x_0)$), $(x_0, f(x_0)) \in P.c(S(f, \alpha))$.
3) $(x_0, f(x_0)) \in Dent(epif) \iff \forall\alpha > f(x_0)$ (or $\exists\alpha > f(x_0)$), $(x_0, f(x_0)) \in Dent(S(f, \alpha))$.
4) $(x_0, f(x_0)) \in S$-$exp(epif) \iff \forall\alpha > f(x_0)$ (or $\exists\alpha > f(x_0)$), $(x_0, f(x_0)) \in S$-$exp(S(f, \alpha))$.

Proof. Only implications ($\Leftarrow$) are not obvious.
1) Let $\alpha > f(x_0)$ such that $(x_0, f(x_0)) \in Ext(S(f, \alpha))$ and suppose that there exists $(a, \lambda) \neq (b, \beta)$ in $epif$ such that $(x_0, f(x_0)) = \frac{1}{2}(a, \lambda) + \frac{1}{2}(b, \beta)$. Necessarily, only one of the two elements (a, λ) or (b, β) is in $S(f, \alpha)$, for $(x_0, f(x_0))$ is extremal in $S(f, \alpha)$. Suppose that $(b, \beta) \notin S(f, \alpha)$, then $f(x_0) \leq \alpha < \beta$. Thus $t = \frac{(\beta-\alpha)}{(\beta-f(x_0))} \in]0, 1[$ and $(z, \alpha) \in S(f, \alpha)$ where $z = tx_0 + (1 - t)b$. Set $r = \frac{1-t}{2-t} \in]0, 1[$. Eliminating b between the relations $z = tx_0 + (1 - t)b$ and $x_0 = \frac{1}{2}a + \frac{1}{2}b$, and β in the relation $\alpha = tf(x_0) + (1 - t)\beta$, we get

$$x_0 = \frac{1-t}{2-t}a + \frac{1}{2-t}z = ra + (1-r)z \text{ and } f(x_0) = \frac{1-t}{2-t}\lambda + \frac{1}{2-t}\alpha = r\lambda + (1-r)\alpha$$

that is $(x_0, f(x_0)) = r(a, \lambda) + (1 - r)(z, \alpha)$; which contradicts the hypothesis $(x_0, f(x_0)) \in Ext(S(f, \alpha))$.
2) Suppose that $(x_0, f(x_0)) \in P.c(S(f, \alpha))$ for some $\alpha > f(x_0)$, and let us show that $(x_0, f(x_0)) \in P.c(epif)$. Let $(x_i, r_i)_{i\in I}$ be a generalized sequence which converges weakly in $epif$ to $(x_0, f(x_0))$. Since $r_i \to f(x_0)$ in $\mathbb{R}$ and $\alpha > f(x_0)$ then, there exists $i_0 \in I$ such that $i \geq i_0 \Rightarrow r_i < \alpha$. We deduce that the family $(x_i, r_i)_{i\geq i_0}$ is in $S(f, \alpha)$ and thus

converges in the norm topology to $(x_0, f(x_0))$.
3) It is a direct consequence of 1), 2) and the following property which is due to [15] "If A is a bounded closed convex set then $Dent(A) = Ext(A) \cap P.c(A)$".
4) Suppose that $(x_0, f(x_0))$ is strongly exposed in $S(f,\alpha)$ with $\alpha > f(x_0)$. Then, by virtue of Corollary 2.3, there exists $x' \in \partial f(x_0)$ such that $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $S(f,\alpha)$. Since $x' \in \partial f(x_0)$, observe that $\langle x', x_0\rangle - f(x_0) = \delta^{\star}_{epif}(x', -1)$. Let $(x_n, r_n)_n$ be a sequence in $epif$ such that

$$\lim_n (\langle x', x_n\rangle - r_n) = \langle x', x_0\rangle - f(x_0). \tag{2}$$

We can suppose that $(r_n)_n$ is bounded and take $M > 0$ such that

$$\alpha < M \text{ and } |r_n| \leq M, \qquad \forall n \in N.$$

We will show that $r_n \to f(x_0)$. Indeed, set $\beta = \frac{\alpha - f(x_0)}{M - f(x_0)}$, $y_n = \beta.x_n + (1-\beta)x_0$ and $s_n = \beta r_n + (1-\beta) f(x_0)$. We have $\beta \in]0,1[$ and

$$\begin{aligned} s_n = \frac{\alpha - f(x_0)}{M - f(x_0)} r_n + \frac{M-\alpha}{M - f(x_0)} f(x_0) &= \frac{M - r_n}{M - f(x_0)} f(x_0) + \frac{r_n - f(x_0)}{M - f(x_0)}\alpha \\ &\leq \frac{M - r_n}{M - f(x_0)}\alpha + \frac{r_n - f(x_0)}{M - f(x_0)}\alpha = \alpha. \end{aligned}$$

Since f is convex, we get $f(y_n) \leq \beta f(x_n) + (1-\beta) f(x_0) \leq \beta.r_n + (1-\beta) f(x_0) \leq \alpha$. Consequently $(y_n, s_n) \in S(f,\alpha)$. It follows that

$$\begin{aligned} \langle x', y_n\rangle - s_n &= \beta\langle x', x_n\rangle + (1-\beta)\langle x', x_0\rangle - \beta.r_n - (1-\beta) f(x_0) \\ &= \beta\left(\langle x', x_n\rangle - r_n\right) + (1-\beta)\left(\langle x', x_0\rangle - f(x_0)\right). \end{aligned}$$

Taking the limit when $n \to +\infty$, and using equality (2), we obtain

$$\lim_n (\langle x', y_n\rangle - s_n) = \langle x', x_0\rangle - f(x_0).$$

Using once again the fact that $(x_0, f(x_0))$ is strongly exposed in $S(f,\alpha)$, we deduce that

$$\lim_n s_n = \lim_n \left(\frac{\alpha - f(x_0)}{M - f(x_0)} r_n + \frac{M - \alpha}{M - f(x_0)} f(x_0) \right) = f(x_0);$$

which implies that

$$\lim_n \left(\frac{\alpha - f(x_0)}{M - f(x_0)} r_n \right) = f(x_0) - \frac{M-\alpha}{M - f(x_0)} f(x_0) = \frac{\alpha - f(x_0)}{M - f(x_0)} f(x_0),$$

and thus, $r_n \to f(x_0)$. Therefore, $r_n \leq \alpha$ for every n large enough, because $f(x_0) < \alpha$ and $(x_n, r_n) \in S(f,\alpha)$, for n large enough. Using once more the fact that $(x_0, f(x_0))$ is strongly exposed in $S(f,\alpha)$, we get finally that $(x_n, r_n)_n$ converges in the norm topology to $(x_0, f(x_0))$ in $S(f,\alpha)$, and by the way in $epif$, which shows that $(x_0, f(x_0))$ is a strongly exposed point in $epif$.■

3. Duality Between Extremality and Differentiability.

For any function $h : X \to]-\infty, +\infty]$, we define the subset $C(h)$ of X' by

$$C(h) = \{x' \in X', \langle x', x\rangle \le h(x), \forall x \in X\}$$

of all continuous linear functions which are less than or equal to h. Phelps ([16], page 78) defines this subset for a semi-norm. Notice that the subset $C(h)$ is $\sigma(X', X)$-closed convex and it is bounded if h takes finite values ($domh = X$). In the particular case where $h = \|.\|$, we have $C(\|.\|) = B_{X'}$, and it is well known that $\partial\|.\|(x) = \{x' \in B_{X'}, \langle x', x\rangle = \|x\|\}$. The latter equality can be generalized to positively homogeneous (not necessarily positive) functions. Furthermore, under some additional assumptions it constitutes a characterization of these functions.

Lemma 3.1. *Let $h : X \to]-\infty, +\infty]$ be a positively homogeneous function. Suppose that $C(h)$ is not empty. Then, for every $x \in X$ we have*

1) $\partial h(x) = \{x' \in C(h), \langle x', x\rangle = h(x)\}$.
2) $h(x) = \delta^\star(x, C(h))$.

Furthermore, if $\partial h(x) \neq \emptyset$ for every $x \in X$, then, h is positively homogeneous if and only if $\partial h(x) = \{x' \in C(h), \langle x', x\rangle = h(x)\}$ for every $x \in X$.

Proof. Suppose that h is positively homogeneous and let $x \in X$. If $\partial h(x) = \emptyset$, then for every $x' \in C(h)$ there exists $y \in X$ such that $\langle x', y\rangle - h(y) \ge \langle x', x\rangle - h(x)$. But $\langle x', y\rangle \le h(y)$. So, we get $\langle x', x\rangle < h(x)$; and consequently $\{x' \in C(h), \langle x,' x\rangle = h(x)\} = \emptyset$, which implies the desired equality. If $\partial h(x) \neq \emptyset$, let $x' \in \partial h(x)$. We have

$$h(ry) - h(x) \ge \langle x', ry - x\rangle, \quad \forall y \in X, \quad \forall r > 0. \tag{3}$$

Dividing this inequality by r and using the fact that h is positively homogeneous, we get $h(y) - \frac{1}{r}h(x) \ge \langle x', y\rangle - \frac{1}{r}\langle x', x\rangle$. Let $r \to +\infty$, then we obtain $h(y) \ge \langle x', y\rangle$ for every $y \in X$. That is $x' \in C(h)$. Replacing y by 0 in (3), we obtain $h(x) \le \langle x', x\rangle$, which implies that $\langle x', x\rangle = h(x)$. For the other inclusion, let $x' \in C(h)$ such that $\langle x', x\rangle = h(x)$. For every $y \in X$ we have $h(y) - h(x) \ge \langle x', y\rangle - h(x) = \langle x', y\rangle - \langle x', x\rangle$ which is equivalent to $x' \in \partial h(x)$.
For the second inequality, notice that $\delta^\star(x, C(h)) \le h(x)$ for every $x \in X$. If there exists $x' \in \partial h(x)$, then $\langle x', x\rangle - h(x) \ge 0$ because h(0) = 0. We get that $\delta^\star(x, C(h)) = h(x)$ for $x' \in C(h)$. Let us check the second assertion. If $x' \in \partial h(x)$, then $\langle x', y\rangle \le h(y)$ for every $y \in X$. In the particular case where $y = \lambda x$, we have $\langle x', \lambda x\rangle \le h(\lambda x)$. Since $\langle x', x\rangle = h(x)$, we deduce that $\lambda h(x) \le h(\lambda x)$. We obtain the other inequality, $h(\lambda x) \le \lambda h(x)$, by taking $x' \in \partial h(\lambda x)$ and using the same method as above.■

Remarks 3.2. *A positively homogeneous function h can not be rotund at any point different from zero. Indeed, recall that the subdifferential at x_0 of such function h is given by*

$$\partial h(x_0) = \{x' \in C(h), \langle x', x\rangle \le h(x) \text{ and } \langle x', x_0\rangle = h(x_0), \forall x \in X\}.$$

The rotundity of h at x_0 implies that there exists $x' \in \partial h(x_0)$ such that for every $\varepsilon > 0$ we can find a real $\delta > 0$ such that $\langle x', x\rangle + \delta \leq h(x)$ whenever $\|x - x_0\| > \varepsilon$.
In the particular case $x = \lambda x_0$, where $\lambda > 0$ and $\|x - x_0\| > \varepsilon$ (which is a possible choice since $x_0 \neq 0$), we will get $\lambda\langle x', x_0\rangle + \delta \leq \lambda h(x_0)$ which contradicts the fact that $\langle x', x_0\rangle = h(x_0)$.

A necessary condition for the (Gateaux or Frechet) differentiability at x_0 of a function h is given by the fact that $\partial h(x_0)$ consists of one point. In the particular case of positively homogeneous function, this condition implies, by virtue of Lemma 3.1, that there exists only one point $x' \in X'$ such that $\langle x', x\rangle = h(x)$. i.e., x is exposed by x' in $C(h)$. In fact, we have the following result.

Corollary 3.3. *Let $h : X \to]-\infty, +\infty]$ be a lower semicontinuous positively homogeneous function, $x_0 \in int(domh)$. Then, h is Gateaux differentiable at x_0 and $\partial h(x_0) = \{x'\}$ if and only if x' is exposed by x_0 in $C(h)$.*

Proof. h is Gateaux differentiable at x_0 if and only if $\partial h(x_0)$ consists at only one point x', which implies that $x' \in C(h)$ and $\langle x', x_0\rangle > \langle y', x_0\rangle$ for every $y' \in C(h)$, that is x' is exposed by x_0 in $C(h)$. Conversely, if x' is exposed by x_0 in $C(h)$ then, we have in particular $\langle x', x_0\rangle \geq \langle y', x_0\rangle$ for every $y' \in \partial h(x_0), (\partial h(x_0) \neq \emptyset$, because $x_0 \in int(domh))$. Since $\langle y', x_0\rangle = h(x_0)$, by virtue of Lemma 3.1, we also have $\langle x', x_0\rangle = h(x_0)$ and then $x' \in \partial h(x_0)$. If there exists $y' \in \partial h(x_0), y' \neq x'$, we will have $\langle x', x_0\rangle > \langle y', x_0\rangle = h(x_0)$ which is absurd. So, $\partial h(x_0) = \{x'\}$ and consequently h is Gateaux differentiable at x_0 and $dh(x_0) = \{x'\}$. ■

In the following, we characterize the Frechet differentiability by the same method used above for the Gateaux differentiability. This characterization is a generalization of a result of Phelps ([16], Proposition 5.11, page 78). First, we recall the well known property ([16], Proposition 2.8, page 19): An application $\rho : X \to X'$ is said to be a selection of the subdifferential ∂g of a function $g : X \to]-\infty, +\infty]$ if $\rho(x) \in \partial g(x)$ for every $x \in dom(\partial g) = \{x \in X,\ \partial g(x) \neq \emptyset\}$. Let g be a convex function which is continuous on a non empty open subset D of X. Then, g is Gateaux (resp. Frechet) differentiable at $x \in D$ if and only if there exists a selection ρ of ∂g which is continuous at x when X is equipped with the norm topology and X' with the $\sigma(X', X)$ topology (resp. the norm topology).

Proposition 3.4. *Let $h : X \to] - \infty, +\infty]$ be a lower semicontinuous positively homogeneous function and $x_0 \in int(domh)$. Then, h is Frechet differentiable at x_0 and $dh(x_0) = \{x'_0\}$ if and only if x'_0 is strongly exposed by x_0 in $C(h)$.*

Proof. Let h be Frechet differentiable at x_0 and that its derivative is given by $dh(x_0) = x'_0$. To check that x'_0 is strongly exposed in $C(h)$ by x_0, it is sufficient to show that all slices $S(C(h), x_0, \alpha)$ contain x'_0 and have a diameter as small as we like when $\alpha \to 0$. Indeed, by virtue of Lemma 3.1, we have $\partial h(x_0) = \{x'_0\}$ and x'_0 is the only point in $C(h)$ which verifies $\langle x_0', x_0\rangle = h(x_0)$; furthermore, $h(x_0) = \delta^\star(x_0, C(h))$. So, for every $\alpha > 0$ we have,

$$\begin{aligned} S(C(h), x_0, \alpha) &= \{x' \in C(h), \langle x', x_0\rangle > \delta^\star(x_0, C(h)) - \alpha\} \\ &= \{x' \in C(h), \langle x', x_0\rangle > h(x_0) - \alpha\}. \end{aligned}$$

It follows that $x_0' \in S(C(h), x_0, \alpha)$ for every $\alpha > 0$. Let $\varepsilon > 0$ be fixed. Since h is Frechet differentiable at x_0, there exists $r > 0$, which can be supposed less than $\frac{\varepsilon}{2}$, such that

$$0 \le h(x_0 + x) - h(x_0) - \langle x_0', x\rangle \le \frac{\varepsilon}{2}\|x\| \text{ for every } x \text{ with } \|x\| \le r.$$

Consider the slice $S(C(h), x_0, r^2)$. For each x' in this slice, we have,

$$|\langle x_0', x_0\rangle - \langle x', x_0\rangle| = |h(x_0) - \langle x', x_0\rangle| < r^2$$

So,

$$\begin{aligned} \langle x', x\rangle - \langle x_0', x\rangle &= \langle x', x_0 + x\rangle - \langle x_0', x_0\rangle - \langle x_0', x\rangle + \langle x_0', x_0\rangle - \langle x', x_0\rangle \\ &\le h(x_0 + x) - h(x_0) - \langle x_0', x\rangle + |\langle x_0', x_0\rangle - \langle x', x_0\rangle| \\ &\le \frac{\varepsilon}{2}\|x\| + r^2, \end{aligned}$$

for every x such that $\|x\| \le r$. We deduce that $|\langle x', x\rangle - \langle x_0', x\rangle| \le \varepsilon$ for every x which verifies $\|x\| \le 1$ and consequently $\|x' - x_0'\| \le \varepsilon$, which shows that the diameter of the slice $S(C(h), x_0, r^2)$ is less than 2ε. Conversely, suppose that x_0' is strongly exposed by x_0 in $C(h)$. Let ρ be a selection of ∂h such that $\rho(x_0) = x_0'$. We must check that ρ is continuous at x_0. Let $(x_n)_{n\ge 1}$ be a sequence in *domh* which converges to x_0. The sequence $(\rho(x_n)_n)$ is bounded in X'. Using Lemma 3.1 we get that $\langle \rho(x_n), x_n\rangle = h(x_n)$ for every n, and we have $\langle \rho(x_n), x_0\rangle = \langle \rho(x_n), x_n\rangle - h(x_0) + \langle \rho(x_n), x_0\rangle - \langle \rho(x_n), x_n\rangle + h(x_0)$. So, $\langle \rho(x_n), x_0\rangle - h(x_0) = h(x_n) - h(x_0) + \langle \rho(x_n), (x_0 - x_n)\rangle$. For every $\varepsilon > 0$, there exists N such that $\langle \rho(x_n), x_0\rangle - h(x_0) \ge \frac{-\varepsilon}{2} - \frac{\varepsilon}{2}$ for $n \ge N$. So, $\langle \rho(x_n), x_0\rangle \ge h(x_0) - \varepsilon = \delta^\star_{C(h)}(x_0) - \varepsilon$, for $n \ge N$. Consequently, $\rho(x_n) \in S(C(h), x_0, \varepsilon)$. Since $\rho(x_0) \in S(C(h), x_0, \varepsilon)$ and the diameter of such slice tends to 0 when $\varepsilon \to 0$, we deduce that $\|\rho(x_n) - \rho(x_0)\| \to 0$. Using the above mentioned result (cf [16]) we get that h is Frechet differentiable at x_0.■

Application. Let $f : X \to]-\infty, +\infty]$ be a convex function. Then, its polar $f^\star$ is convex too and $\sigma(X', X)$-lower semicontinuous and $epif^\star$ is a $\sigma(X', X)$ closed convex set. The function $h(x,t) = \delta^\star_{epif^\star}(x,t)$ is positively homogeneous and convex. Using the Hahn-Banach theorem we get

$$C(h) = \{(x', \lambda) \in X' \times \mathbb{R}, \langle (x', \lambda), (x,t)\rangle \le \delta^\star_{epif^\star}(x,t), \quad \forall (x,t) \in X \times \mathbb{R}\} = epif^\star$$

Consequently, by virtue of Lemma 3.1, which characterize the subdifferential of a positively homogeneous function, we have

$$\begin{aligned} \partial h(x,t) &= \{(x', \lambda) \in C(h), \langle (x', \lambda), (x,t)\rangle = h(x,t)\} \\ &= \{(x', \lambda) \in epif^\star, \langle x', x\rangle + \lambda.t = \delta^\star_{epif^\star}(x,t)\}. \end{aligned}$$

In the particular case where $t=-1$ we get,

$$\begin{aligned}\partial h(x,-1) &= \{(x',\lambda)\in epif^\star, \langle x',x\rangle-\lambda=\delta^\star_{epif^\star}(x,-1)\}\\ &= \{(x',\lambda), f^\star(x')\le\lambda, \quad \lambda=\langle x',x\rangle-f(x)\}\\ &= \{(x',\lambda), \langle x',x\rangle-f(x)=\lambda\ge f^\star(x')\}\\ &= \{(x',f^\star(x')), x'\in\partial f(x)\}\end{aligned}$$

By the same method, we check that $\partial h(-tx,t)=\partial h(x,-1)$, and $\partial h(x,t)=\{(x',f^\star(x')), x'\in\partial f(\frac{x}{-t})\}$, for every $t<0$. Now, we check the relation between the selections of ∂f and ∂h. Let ψ be a selection of ∂h. In particular, we have $\psi(x,-1)\in\partial h(x,-1)$ for every x such that $(x,-1)\in domh$. So, for this x there exists $x'\in\partial f(x)$ such that $\psi(x,-1)=(x',f^\star(x'))$. If we set $\rho(x)=x'$, then ρ is a selection of ∂f and we have $\psi(x,-1)=(\rho(x),f^\star(\rho(x)))$ for every x such that $(x,-1)\in domh$. Conversely, if ρ is a selection of ∂f then, $(\rho(x),f^\star(\rho(x)))\in\partial h(x,-1)$ for every $x\in domf$.

Using the same considerations as Proposition 3.4, we can give a characterization of the differentiability of a convex function by the differentiability of the support function of the epigraph of its polar. Hence, we bring back the study of the differentiability of convex functions to positively homogeneous ones.

Theorem 3.5. *Let $f: X\to]-\infty,+\infty]$ be a lower semicontinuous proper convex function and $h(.,.)=\delta^\star_{epif^\star}(.,.)$. Then, f is Frechet (resp. Gateaux) differentiable at $x_0\in int(domf)$ if and only if h is Frechet (resp. Gateaux) differentiable at $(x_0,-1)$.*

Proof. Let τ denote indifferently the norm or the $\sigma(X',X)$ topology of X'. If f is Frechet (resp. Gateaux) differentiable at x_0, then there exists a selection of ∂f, say $\rho: dom(\partial f)\to X'$, which is continuous at x_0 when $dom(\partial f)$ is equipped with the norm topology and X' with τ. Let ψ be a selection of ∂h such that $\psi(x,-1)=(\rho(x),f^\star(\rho(x)))$ for every $x\in domf$. Let us show that $\psi:(dom(\partial h),\|.\|)\to X'\times\mathbb{R}$ is continuous at x_0 when $X\times\mathbb{R}$ is equipped with the product topology of τ and the usual topology of $\mathbb{R}$. Let $(x_i,t_i)_{i\in I}$ be a generalized sequence in $domh$ which converges in norm to $(x_0,-1)$ (when τ is the norm topology, we can take a sequence). Without loss of generality, we can suppose that $t_i<0$ for every i, since, $\lim_{i\to+\infty} t_i=-1$. By virtue of the choice ψ, it follows that $\psi(x_i,t_i)=(x'_i,f^\star(x'_i))$, where $x'_i=\rho(\frac{x_i}{-t_i})\in\partial f(\frac{x_i}{-t_i})$, for every i. The continuity of ρ implies that $(x'_i)_i$ converges to $\rho(x_0)$ with respect to τ. Using the equality $\langle x'_i,\frac{x_i}{-t_i}\rangle-f(\frac{x_i}{-t_i})=f^\star(x'_i)$, we deduce that $f^\star(x'_i)$ converges to $f^\star(\rho(x_0))$. Consequently,

$$\lim_{i\to+\infty}\psi(x_i,t_i)=\lim_i(x'_i,f^\star(x'_i))=(\rho(x_0),f^\star(\rho(x_0)))=\psi(x_0,-1).$$

This implies that ψ is continuous at $(x_0,-1)$ with respect to the given topologies. Then, h is Frechet (resp. Gateaux) differentiable at $(x_0,-1)$. So, there exists a selection $\psi: dom(\partial h)\to X'\times\mathbb{R}$ of ∂h which is continuous at $(x_0,-1)$ with respect to the given topologies. In particular, for every $x\in dom(\partial f)$, we have $\psi(x,-1)\in\partial h(x,-1)$. Hence, there exists $x'\in\partial f(x)$ such that $\psi(x,-1)=(x',f^\star(x'))$. Consider the function

$\rho : (dom(\partial f), \|.\|) \longmapsto (X', \tau)$ defined by $\rho(x) = x'$. we check that ρ is a selection of ∂f and is continuous at x_0. Indeed, let $(x_i)_{i\in I}$ be a generalized sequence which converges to x_0. Since ψ is continuous at $(x_0, -1)$, we have $(\rho(x_0), f^\star(\rho(x_0))) = \psi(x_0, -1) = \lim_{i\to+\infty} \psi(x_i, -1) = \lim_{i\to+\infty} (\rho(x_i), f^\star(\rho(x_i)))$ which is equivalent to $\lim_{i\to+\infty} \rho(x_i) = \rho(x_0)$ in (X', τ) and $\lim_{i\to+\infty} f^\star(\rho(x_i)) = f^\star(\rho(x_0))$. Then, ρ is continuous at x_0 when X' is equipped with τ. Afterwards, f is Frechet (resp. Gateaux) differentiable when $\tau = \|.\|$ (resp. $\tau = \sigma(X', X)$).■

Corollary 3.6. *Let $f : X \to]-\infty, +\infty]$ be a lower semicontinuous proper convex function and $x_0 \in int(domf)$. Then, f is Frechet (resp. Gateaux) differentiable at x_0 and its derivative is $df(x_0) = x'$ if and only if $(x', f^\star(x'))$ is strongly exposed (resp. exposed) by $(x_0, -1)$ in $epif^\star$.*

Proof. Apply Theorem 3.5 and Proposition 3.4 (resp. Corollary 3.3) to the function $h = \delta^\star_{epif^\star}$.■

If we combine Corollary 3.6 and Proposition 2.4, we obtain a result which is due to Asplund-Rockafellar ([2], Theorem 1, p:445).

Corollary 3.7. *A lower semicontinuous convex function $f : X \to]-\infty, +\infty]$ is Frechet (resp. Gateaux) differentiable at $x_0 \in int(domf)$ if and only if its polar $f^\star$ is rotund (resp. strictly convex) at $df(x_0)$.*

Proof. By virtue of Corollary 3.6, f is Frechet (resp. Gateaux) differentiable at x_0 if and only if there exists $x' \in \partial f(x_0)$ such that $(x', f^\star(x'))$ is strongly exposed (resp. exposed) in $epif^\star$ by an element of $X \times \mathbb{R}$. This is equivalent, by virtue of Proposition 2.4, to $f^\star$ is rotund (resp. strictly convex) at $df(x_0)$.■

4. Application to the Radon-Nikodym Property.

In this section, we use the properties of lower semi-continuous convex functions obtained in the previous sections to formulate a new characterization of Banach spaces with the Radon-Nikodym property; using geometrical properties of convex functions defined on Banach spaces. First, we recall some definitions.

• A point $x_0 \in X$ is said to be a strong minimum of f if the following assertions hold:

i) $f(x_0) \le f(x), \forall x \in X$

ii) $\|x_n - x_0\| \to 0$ whenever $f(x_n) \to f(x_0)$, for each sequence $(x_n)_n$ in X.

• A Banach space is said to have the Radon-Nikodym property (R.N.P) if every nonempty closed bounded subset of X has at least one denting point. There are several others characterizations of the R.N.P (see, [4], [6], [10]).

In the following, we improve Proposition 2.4 by adding a third assertion concerning the strong minimum.

Proposition 4.1. *Let $f : X \to]-\infty, +\infty]$ be a convex lower semi-continuous function and $x_0 \in domf$. The following statements are equivalent:*

1) *There exists $x' \in \partial f(x_0)$ such that x_0 is a strong minimum of $f - x'$.*
2) *There exists $x' \in X'$ such that $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $epif$.*
3) *f is rotund at x_0.*

Proof. According to Proposition 2.4, it remains to prove the implications 1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 1). For the first one, let $x' \in \partial f(x_0)$ be such that $f - x'$ has a strong minimum at x_0. Let $\varepsilon > 0$ and $r \leq min(\frac{\varepsilon}{2\|x'\|}, \varepsilon)$. Using the second assertion of the strong minimum's definition, there exists $\delta > 0$ such that

$$|f(x) - \langle x', x\rangle - f(x_0) + \langle x', x_0\rangle| \leq \delta \Rightarrow \|x - x_0\| \leq r.$$

Notice that the slice $S(epif, (x', -1), \alpha)$ has a diameter less than ε, provided that $\alpha \leq min(\frac{\varepsilon}{2}, \delta)$. Let $(x_0, f(x_0))$ be strongly exposed by $(x', -1)$ in $epif$, $\varepsilon > 0$ be fixed and $\alpha > 0$ such that the slice $S(epif, (x', -1), \alpha)$ has a diameter less than ε. Since every pair $(x, f(x))$ with $f(x) < \langle x', x\rangle - f(x_0) - \langle x', x_0\rangle + \alpha$ is in $S(epif, (x', -1), \alpha)$, we deduce that $\|x - x_0\| \leq \varepsilon$; which shows that f is rotund at x_0. For the other implication, notice that x' which appears in the definition of rotundity, is in $\partial f(x_0)$. The condition ii) follows by an easy calculation from the definition of rotundity.■

The following theorem, the aim of this section, gives a new characterization of the Radon-Nikodym property. Another aspect has been examined in ([22], Corollary 3.2).

Theorem 4.2. *A Banach space X has the R.N.P if and only if every convex lower semi-continuous and bounded below function on X verifies one of the three statements of Proposition 4.1 .*

Proof. We prove that the R.N.P is equivalent to the existence of a point of rotundity. Let C be a bounded closed convex subset of X. Its indicator functions $\delta_C(.)$ is lower semicontinuous proper convex and bounded below. Then, it admits a point of rotundity $x_0 \in C$. Thus, for every $\varepsilon > 0$, there exists $x' \in X'$ and a real $r > 0$ such that

$$\langle x', x_0\rangle < \langle x', x\rangle + r \quad \Rightarrow \quad \|x - x_0\| \leq \varepsilon.$$

Using the fact that $\langle x', x_0\rangle = \delta_C^\star(x')$, we deduce that the slice $S(C, x', r)$ contains x_0 and its diameter is less than ε, which means that $x_0 \in Dent(C)$; and by the way the condition is sufficient. To see that it is a necessary one, we, use Corollary 2.3 and Proposition 2.6. Let $f : X \to]-\infty, +\infty]$ be a lower semicontinuous convex function which is bounded below, and $\alpha > inf\{f(x), x \in X\}$. We will show that the set $\{(x, \lambda), f(x) \leq \lambda \leq \alpha\}$ admits a strongly exposed point $(x_0, f(x_0))$ with $f(x_0) < \alpha$. We then conclude, by virtue of Proposition 2.6 and Proposition 2.4, that f admits a rotundity point. The hypothesis on f imply that $S(f, \alpha)$ is a bounded closed convex nonempty subset of $X \times \mathbb{R}$. Since $X \times \mathbb{R}$ has the Radon-Nikodym Property, the set $S\text{-}exp(S(f, \alpha))$ is nonempty and we have $S(f, \alpha) = \overline{co}(S\text{-}exp(S(f, \alpha)))$ ([4] , [17]). If all elements $(x, f(x))$ of $S\text{-}exp(S(f, \alpha))$ are such that $f(x) = \alpha$, we will have $S(f, \alpha) \subset \overline{co}((x, \alpha), f(x) = \alpha) = \{(x, \alpha), f(x) \leq \alpha\}$

which is absurd because with the choice of α, there exists $x \in X$ such that $f(x) < \alpha$.■

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