FUNCTIONALS MEASURE AND INF-COMPACTNESS OF INTEGRAL FUNCTIONALS

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Abstract. We give some disjunction type results for measures and a property of domain of functional measure and we applied them to characterize by growth conditions, the inf-compactness property of an integral functional $I_f(.,.) = \int\limits_{\Omega} f(.,.,.)d\mu$ defined on the product of two integral spaces.

Introduction

Inf-compactness and lower semicontinuity (lsc) properties of integral functionals, play a crucial role in several Mathematics problems (Variational and hemivariational inequalities, optimization of forms, optimal control, Mathematical economics, statistics,$\cdots$). Those properties have been studied by several authors (E.J.Balder [2], A.Bourass, B.Ferrahi, O.Kahlaoui [5], G.Bottaro, P.Oppezi [3], C.Castaing, P.Clausure [6], L.Césari [8],[9], I.Ekeland, R.Temam [10], E.Giner [12], A.D.Ioffé [13], C.Olech [14],[15],[16], Rockafellar [17]) and others.

In this paper, we are interested by the study of inf-compactness properties of an integral functional with two variables defined as follows: $I_f(u,v) = \int\limits_{\Omega} f(\omega, u(\omega), v(\omega))d\mu$, f is an integrand defined on $\Omega \times X \times Y$, where Ω is a measurable space and X, Y are Banach spaces, the functions u and v run over integrals spaces. To our knowledge, it is the first time that the problem of the inf-compactness of an integral functional defined on a product of spaces is studied. In the first paragraph, we present some technical "disjunction type" results for a family of measures and we establish a property of the domain of functional measures where the σ-decomposability plays a key role. In the second paragraph we give growth conditions that insure the inf-compactness of I_{f_u} for all u in a decomposable set and we also give a characterization in order to the integral functional associated to $\inf\limits_{x\in\Gamma(\omega)} f(\omega,x,y)$ be inf-compact. At last, conditions on the integrand f will be introduced in order to obtain the inf-compactness of I_f on all a neighborhood of $u_0 \in L^0_X(\Omega,\mu)$ as soon as it is inf-compact in u_0.

In the sequel, X and Y denote two separable Banach spaces, $\mathcal{B}_X$, $\mathcal{B}_Y$ their borel tribes respectively and $(\Omega, \mathcal{A}, \mu)$ a measurable space such that μ is an atomless complete finite non negative measure. We note $L^0_X(\Omega,\mu)$ (resp. $L^1_X(\Omega,\mu)$) (resp.$L^\infty_X(\Omega,\mu)$) the space of all measurable functions (resp. Bochner integrable functions) (resp. essentially bounded functions) from Ω into X.
Let $f : \Omega\times X\times Y \to \overline{\mathbb{R}}$ be an integrand (i.e.: $\mathcal{A}\otimes\mathcal{B}(X)\otimes\mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}})$-measurable), which is not everywhere equal to $+\infty$. Observe that $\omega \to f(\omega, u(\omega), v(\omega))$ is also measurable for every $u \in L^0_X(\Omega,\mu)$ and every $v \in L^0_Y(\Omega,\mu)$. For every $x \in X$, f_x denotes the function defined for each $(w, y) \in \Omega \times Y$ by $f_x(\omega, y) = f(\omega, x, y)$. The conjugate function or polar of f_x with respect to the (Y, Y') duality is defined for each $(\omega, y') \in \Omega \times Y'$ by $f^*_x(\omega, y') = \sup\limits_{y\in Y}(< y, y' > -f_x(\omega, y))$. For every $u \in L^0_X(\Omega,\mu)$, set f_u or $f(u, .)$ the function defined for each $(\omega, y) \in \Omega \times Y$ by $f(\omega, u(\omega), y)$, and $f^*_u(\omega, y') = \sup\limits_{y\in Y}(< y, y' > -f(\omega, u(\omega), y))$ its polar with respect to the (Y, Y') duality. We define the integral functional associated to f as follows: $I_f : L^1_X(\Omega,\mu) \times L^1_Y(\Omega,\mu) \to \overline{\mathbb{R}}$ with

$$I_f(u,v) = \begin{cases} \int\limits_\Omega f(\omega, u(\omega), v(\omega))d\mu & \text{if} \quad f(., u(.), v(.)) \in L^1{}_{\mathbb{R}}(\Omega,\mu) \\ +\infty & \text{if not} \end{cases}$$

- We say that a function $g : X \times Y \to \overline{\mathbb{R}}$ is $(\|.\|, \sigma)$-lsc if it is lower semicontinuous on $X\times Y$ when X is equipped with the norm topology and Y with the weak topology $\sigma(Y, Y')$. The function g is proper if it takes no $-\infty$ values and not everywhere equal to $+\infty$.
- We say that a function $g : E \to \overline{\mathbb{R}}$ is τ inf-compact, where τ is a topology on E, if the level sets of g (i.e.: $\{x \in E, \ \ g(x) \le M\}$, for some scalar M) are compacts with respect to τ.
- For every coercive even non negative normal convex integrand which is continuous and takes zero in the origin (Young function) $\varphi : \Omega \times X \to \overline{\mathbb{R}}_+$, set:
  - C_φ: The Orlicz class associated to φ, that is the domain of I_φ:

$$C_\varphi = \left\{ u \in L^0_X(\Omega,\mu), \int\limits_\Omega \varphi(\omega, u(\omega))d\mu < +\infty \right\}.$$

  - L_φ: The vector space generated by C_φ that is $\mathbb{R}C_\varphi$.
  - E_φ: The biggest vector subspace contained in C_φ, which exists since $0 \in C_\varphi$.
  - $\|.\|_\varphi$: The Gauge of the convex set $\{I_\varphi(.) \le 1\}$, it defines a norm in L_φ.

- The polar of φ is defined by $\varphi^*(\omega, x') = \sup_{x \in X} (< x, x' > -\varphi(\omega, x))$, it is also a Young function.
- We say that φ satisfies a Δ_2-condition if there exists $\lambda > 1$, $K > 0$ and $a(.) \in L^1{}_{\mathbb{R}^+}(\Omega, \mu)$ such that for almost every $\omega \in \Omega$ and every $x \in X$ we have

$$\varphi(\omega, \lambda x) \leq K\varphi(\omega, x) + a(\omega)$$

- A σ-finite μ-recouvrement of Ω is a non decreasing family $(\Omega_n)_n$ of pairwise disjoint measurable subsets such that $\mu(\Omega \backslash \bigcup_n \Omega_n) = 0$ and $\mu(\Omega_n) < +\infty$ for every n (we shall write $\Omega_n \nearrow \Omega$).
- A μ-partition of Ω is a family $(\Omega_n)_n$ of pairwise disjoint measurable subsets such that $\mu(\Omega \backslash \bigcup_n \Omega_n) = 0$.
- We say that a subset D of $L^0_X(\Omega, \mu)$ is decomposable if for every u_1 and u_2 in D and every measurable subset E, $u_1.\chi_E + u_2.\chi_{E^c}$ is in D, where χ_E denotes the characteristic function of E that is $\chi_E(\omega) = 1$ if $\omega \in E$ and $\chi_E(\omega) = 0$ if not. We say that D is σ-decomposable if for every $(u_n)_n$ in D, and every μ-partition $(\Omega_n)_n$ of Ω, the function $\sum_n u_n.\chi_{\Omega_n}$ is in D. A subset H is rich in D if for every $u \in D$, there exists a sequence $(u_n)_n$ in H and $\Omega_n \nearrow \Omega$ such that $u_n.\chi_{\Omega_n} = u.\chi_{\Omega_n}$ μ-almost everywhere for every $n \in \mathbb{N}$.
- Let $\Gamma : \Omega \to X \times Y$ be a measurable multiapplication (see [7]), set L^0_Γ its profile (i.e.: all $(\mathcal{A}, \mathcal{B}(X) \otimes \mathcal{B}(Y))$-measurable functions $u : \Omega \to X \times Y$ such that $u(\omega) \in \Gamma(\omega)$ μ-almost everywhere.)
- Let f and g be two integrands defined on $\Omega \times Y$ into $\overline{\mathbb{R}}$. We say in accordance with [12] that "g is almost essentially dominated by f" if the following condition holds:
$\forall \lambda > 0 \ \exists b_\lambda > 0, \ \exists a_\lambda \in L^1{}_{\mathbb{R}_+}(\Omega, \mu)$ such that for μ-almost every ω in Ω, we have:

$$g(\omega, \lambda y) \leq b_\lambda f(\omega, y) + a_\lambda(\omega) \quad \forall y \in Y$$

If $b_\lambda = 1$ for every $\lambda > 0$, then we say that "g is essentially dominated by f".

1. Disjunction type results and functional measures

In the following we establish some new disjunction type results. Those results are related to measures, measurable functionals and integrands.

Lemma 1.1. *Let $(\mu_n)_n$ be a sequence of atomless finite non negative measures such that*

$$\mu_n(\Omega) \geq 2^n \qquad \forall n \in \mathbb{N}.$$

Then, there exists a subsequence $(\mu_{n_k})_k$ *and a family of pairwise disjoint measurable subsets* $(A_p)_p$ *such that*

$$\mu_{n_k}(A_k) = 1 \qquad \forall k \in \mathbb{N}.$$

Proof. The construction is based on the following recurrent procedure, we can extract a subsequence $(\mu_{n_k})_k$ such that for every $p \in \mathbb{N}$, there exists a family $\{A_i, \quad i = 1, \cdots, p\}$ of pairwise disjoint measurable subsets which satisfy

$$\mu_{n_k}(A_k) = 1 \qquad k = 1, \cdots, p$$
$$\mu_{n_k}(\Omega \backslash \bigcup_{i=1}^{p} A_i) \geq 2^{n_k - p} \qquad k = p+1, \cdots$$

For $p = 1$, we have $\mu_1(\Omega) \geq 2$ and μ_1 is atomless, then there exists a measurable subset B_1 such that $\mu_1(B_1) = 1$. For every $n \in \mathbb{N}$, we have $\mu_n(\Omega) = \mu_n(B_1) + \mu_n(\Omega \backslash B_1) \geq 2^n$, so $\mu_n(B_1) \geq 2^{n-1}$ or $\mu_n(\Omega \backslash B_1) \geq 2^{n-1}$. Two possibilities present themselves:
1) We can extract a subsequence $(\mu_{n_k})_k$ such that $\mu_{n_k}(\Omega \backslash B_1) \geq 2^{n_k - 1}$ for every $k \in \mathbb{N}$, here $\mu_{n_1} = \mu_1$. Then, we take $A_1 = B_1$ and continue the procedure with $(\mu_{n_k})_k$.
2) There exists $n_2 \in \mathbb{N}$ such that $\mu_n(B_1) \geq 2^{n-1}$ for every $n \geq n_2$. We have $\mu_1(\Omega \backslash B_1) \geq 1$ and μ_1 is an atomless measure. It is sufficient to choose $A_1 \subset \Omega \backslash B_1$ such that $\mu_1(A_1) = 1$ and continue with the following sequence $(\mu_n)_{1, n_2, n_2+1, \dots}$.
Thus, in all cases and by noting, $(\mu_{n_k})_k$ the extracted subsequence, we have:
For $k = 1 \quad \mu_{n_1}(A_1) = 1$
For $k \geq 2 \quad \mu_{n_k}(\Omega \backslash A_1) = \begin{cases} \mu_{n_k}(\Omega \backslash B_1) \geq 2^{n_k - 1} & \text{first case} \\ \mu_{n_k}(\Omega \backslash A_1) \geq \mu_{n_k}(B_1) \geq 2^{n_k - 1} & \text{second case, since } n_k \geq n_2 \end{cases}$

Which completes the construction for p=1. Suppose that we can extract successive subsequence $(\mu_{n_k})_k$ and build a family $A_1, \cdots, A_p$ of pairwise disjoints measurable subsets such that

$$\mu_{n_k}(A_k) = 1 \qquad k = 1, \cdots, p$$
$$\mu_{n_k}(\Omega \backslash \bigcup_{i=1}^{p} A_i) \geq 2^{n_k - p} \quad k = p+1, \cdots$$

Let us show that the property is still true for the following rank $p + 1$. We have $\mu_{n_{p+1}}(\Omega \backslash \bigcup_{i=1}^{p} A_i) \geq 2^{n_{p+1} - p} \geq 2$, and $\mu_{n_{p+1}}$ atomless, there exists $B_{p+1} \subset \Omega \backslash \bigcup_{i=1}^{p} A_i$ such that:

$$\mu_{n_{p+1}}(B_{p+1}) = 1$$

For $k \geq p+2$, we have:

$$\mu_{n_k}(\Omega\backslash\bigcup_{i=1}^{p}A_i) = \mu_{n_k}((\Omega\backslash\bigcup_{i=1}^{p}A_i)\backslash B_{p+1}) + \mu_{n_k}(B_{p+1}) \geq 2^{n_k-p}$$

Consequently, we have one of the following alternative cases.

$$i)\ \ \mu_{n_k}(B_{p+1}) \geq 2^{n_k-p-1} \qquad ii)\ \ \mu_{n_k}((\Omega\backslash\bigcup_{i=1}^{p}A_i)\backslash B_{p+1}) \geq 2^{n_k-p-1}$$

As above we distinguish two cases:

1) There exists an infinite number of indices (k) such that: $\mu_{n_k}((\Omega\backslash\bigcup_{i=1}^{p}A_i)\backslash B_{p+1}) \geq 2^{n_k-p-1}$ and we can extract a subsequence $(\mu_{n_{k_l}})_l$ which verifies:

$$\mu_{n_{k_l}}((\Omega\backslash\bigcup_{i=1}^{p}A_i)\backslash B_{p+1}) \geq 2^{n_{k_l}-p-1}$$

So, we set $A_{p+1} = B_{p+1}$ and we continue the procedure with the following subsequence: $\mu_{n_1}, \cdots, \mu_{n_{p+1}}, (\mu_{n_{k_l}})_l$.

2) There exists $n_{k_0} \in \mathbb{N}$ such that for every $k \geq k_0$ we have :

$$\mu_{n_k}(B_{p+1}) \geq 2^{n_k-p-1}$$

Since:

$$\begin{aligned}\mu_{n_{p+1}}((\Omega\backslash\bigcup_{i=1}^{p}A_i)\backslash B_{p+1}) &= \mu_{n_{p+1}}(\Omega\backslash\bigcup_{i=1}^{p}A_i) - \mu_{n_{p+1}}(B_{p+1}) \\ &\geq 2^{n_{p+1}-p} - 1 \geq 1\end{aligned}$$

and the measure $\mu_{n_{p+1}}$ is atomless, then there exists $A_{p+1} \subset (\Omega\backslash\bigcup_{i=1}^{p}A_i)\backslash B_{p+1}$ such that $\mu_{n_{p+1}}(A_{p+1}) = 1$. We continue with the following subsequence:

$$\mu_{n_1}, \cdots, \mu_{n_{p+1}}, (\mu_{n_k})_{k\geq k_0}$$

Thus, in the two cases we can build a subsequence $(\mu_{n_l})_l$ and a family $A_1, \cdots, A_{p+1}$ of pairwise disjoints measurable subsets such that:

$$\begin{array}{ll}\mu_{n_l}(A_l) = 1 & 1 \leq l \leq p+1 \\ \mu_{n_l}(\Omega\backslash\bigcup_{i=1}^{p+1}A_i \geq 2^{n_l-(p+1)} & l \geq p+2 \quad \blacksquare\end{array}$$

Proposition 1.2. *Let $L \subset L^0_X(\Omega,\mu)$ and $M \subset L^0_Y(\Omega,\mu)$ be two vector subspaces such that L is stable by trancature (i.e.: $u.\chi_A \in L$ if $u \in L$ and $A \in \mathcal{A}$). Let D be a σ-decomposable subset of L and $F : L \times M \to \overline{\mathbb{R}}$ a functional such that:*

i) *The application* $A \longmapsto F(u.\chi_A, v)$ *is* σ*-additive i.e.:* $F(u.\chi_{\bigcup_{i=1}^{\infty} A_i}, v) = \sum_{i=1}^{\infty} F(u.\chi A_i, v)$ *for every* $(u,v) \in dom_D F = \{(u,v) \in D \times M,\ \ F(u,v) < +\infty\}$ *;*

ii) *There exists* $(u_0, v_0) \in D \times M$ *such that* $F(u_0, v_0) < +\infty$.

Then,

$$\bigcap_{u \in D} dom F(u,.) = dom\left(\sup_{u \in D} F(u,.)\right).$$

Proof. Without loss of generality we can suppose that $0 \in D$ and $F(0,0) = 0$. Indeed, let (u_0, v_0) such that $F(u_0, v_0) < +\infty$. Set $D_0 = D - u_0$ and $F_0(u,v) = F(u + u_0, v + v_0) - F(u_0, v_0)$. We have

$$\bigcap_{u \in D} dom F(u,.) = \bigcap_{u' \in D_0} dom F_0(u',.) + v_0. \tag{1}$$

Indeed, let $v = v_1 - v_0 \in \bigcap_{u \in D} dom F(u,.) - v_0$. For every $u' \in D_0$, there exists $u \in D$ such that $u' = u - u_0$. So

$$\begin{aligned} F_0(u', v) &= F(u' + u_0, v + v_0) - F(u_0, v_0) \\ &= F(u, v_1) - F(u_0, v_0) < +\infty \end{aligned}$$

Then $\bigcap_{u \in D} dom F(u,.) \subset \bigcap_{u' \in D_0} dom F_0(u',.) + v_0$. For the other inclusion, let $v = v_1 + v_0 \in \bigcap_{u' \in D_0} dom F_0(u',.) + v_0$. For every $u \in D$, there exists $u' \in D_0$ such that $u = u' + u_0$. Therefore,

$$\begin{aligned} F(u, v) &= F(u' + u_0, v_1 + v_o) \\ &= F_0(u', v_1) + F(u_0, v_0) < +\infty. \end{aligned}$$

Which implies the desired inclusion. The following equality can be shown by using the same method

$$dom\left(\sup_{u \in D} F(u,.)\right) = dom\left(\sup_{u' \in D_0} F_0(u',.)\right) + v_0. \tag{2}$$

Suppose from now on that $0 \in D$ and $F(0,0) = 0$. The inclusion $dom\left(\sup_{u \in D} F(u,.)\right) \subset \bigcap_{u \in D} dom F(u,.)$ is obvious. Let us suppose that the other inclusion does not hold, then there exists $v' \in M$ such that for every $u \in D$ we have $F(u, v') < +\infty$ and $\sup\limits_{u \in D} F(u, v') = +\infty$. Thus, there exists a subsequence $(u_n)_n$ in D such that:

$$F(u_n, v') \geq 2^n \quad \forall n \in \mathbb{N}.$$

For every $A \in \mathcal{A}$ set,

$$\mu_n(A) = F(u_n.\chi_A, v').$$

For every $n \in \mathbb{N}$, μ_n is an atomless measure (hypothesis i)) because it is absolutely continuous by respect to μ. Moreover,

$$\mu_n(\Omega) = F(u_n.\chi_\Omega, v') = F(u_n, v') \geq 2^n \quad \forall n \in \mathbb{N}.$$

Using lemma (1.1), there exists a subsequence $(\mu_{n_k})_k$ of $(\mu_n)_n$ and a family $(A_k)_k$ of pairwise disjoints measurable sets such that for every $k \in \mathbb{N}$ we have $\mu_{n_k}(A_k) = 1$. The element $\bar{u} = \sum\limits_k u_{n_k}.\chi_{A_k}$ is in D because it is σ-decomposable. Observing that $\bar{u}.\chi_{A_k} = u_{n_k}.\chi_{A_k}$, we have

$$\begin{aligned} F(\bar{u}, v') &= F(\sum_k u_{n_k}.\chi_{A_k}, v') = F(\sum_k \bar{u}.\chi_{A_k}, v') \\ &= F(\bar{u}.\sum_k \chi_{A_k}, v') = \sum_k F(\bar{u}.\chi_{A_k}, v') \\ &= \sum_k F(u_{n_k}.\chi_{A_k}, v') = \sum_k \mu_{n_k}(A_k) = +\infty \end{aligned}$$

Which contradicts the fact that $F(u, v') < +\infty$ for every $u \in D$ and completes the proof. ■

As we can show it later, the assertion *i)* in proposition (1.2) can be obtained by using a lower semicontinuity property. Let us consider the following hypothesis. Let L be a semi-normed vector topological space, τ its topology, such that (L, τ) is topologically decomposable ([12], definition 1.1.3) and satisfies the following fundamental hypothesis.

$(F.H)$ The application $(u, A) \in L \times \mathcal{A} \longmapsto u.\chi_A \in L$ is separately continuous (it is sufficient to check it at $(0, \emptyset)$) where L is equipped by τ and $\mathcal{A}$ by the so-called Nikodym topology which is defined as follows:

$$\begin{aligned} A_n \xrightarrow{\mathcal{N}} A \quad &\Leftrightarrow \quad \forall B \in \mathcal{A} \;\; \mu(B) < +\infty \quad \mu(B \cap (A_n \Delta A)) \to 0 \\ &\Leftrightarrow \quad \chi_{A_n} \overset{L^0_X(\Omega,\mu)}{\longrightarrow} \chi_A. \end{aligned}$$

Examples of topological spaces where the $(F.H)$ holds:

- $L^0_X(\Omega, \mu)$ equipped with the topology of convergence in measure ;

- L_φ equipped with the weak topology $\sigma(L_\varphi, L_{\varphi^*})$ when φ is continuous in 0 ;
- E_φ equipped with the norm $\|.\|_\varphi$ topology when $\bar{\varphi}(\omega, t) := \sup\limits_{\|x\|=t} \varphi(\omega, x)$, is finite.

Lemma 1.3. *Let $F : L \times M \to \overline{I\!R}$, and D a τ-closed decomposable subset of L, which contains zero. Suppose that L verifies the fundamental hypothesis $(H.F)$, and that for every $v \in M$, the following conditions hold:*

i) *The application $A \longmapsto F(u.\chi_A, v)$ is additive for every $(u, v) \in domF$ (i.e.: For every finite family $(A_i)_{1\leq i\leq n}$ of pairwise disjoints measurable subsets, we have:*

$$F(u.\chi_{\bigcup\limits_{i=1}^{n} A_i}, v) = \sum_{i=1}^{n} F(u.\chi_{A_i}, v)$$

ii) *The functional $F(., v)$ is proper τ-lsc on D.*

Then, the application $A \longmapsto F(u.\chi_A, v)$ is σ-additive, for every $(u, v) \in D \times M$.

Proof. Let $v \in M$, $u \in D$ and $(A_n)_n$ be a family of pairwise disjoints measurable subsets. we must check that :

$$F(u.\chi_{\bigcup\limits_{n} A_n}, v) = \lim_{n\to+\infty} \sum_{p=0}^{n} F(u.\chi_{A_p}, v).$$

From $i)$, it follows that for every $n \in I\!N$,

$$F(u.\chi_{\bigcup\limits_{n} A_n}, v) = \sum_{p=0}^{n} F(u.\chi_{A_p}, v) + F(u.\chi_{B_n}, v)$$

where $B_n = \bigcup\limits_{p>n} A_p$. Note that B_n is a non-increasing sequence and that $\bigcap\limits_{n} B_n = \emptyset$, which implies $B_n \xrightarrow{\mathcal{N}} \emptyset$.
We deduce that

$$\tau - \lim_{n\to+\infty} u.\chi_{B_n} = 0 \qquad \text{and} \qquad \tau - \lim_{n\to+\infty} \sum_{p=0}^{n} u.\chi_{A_p} = u.\chi_A,$$

where $A = \bigcup\limits_{n} A_n$. The functional F is proper τ-lsc, then,

$$0 = F(0, v) \leq \liminf_{n\to+\infty} F(u.\chi_{B_n}, v)$$

and

$$\begin{aligned} -\infty < F(u.\chi_A, v) &\leq \liminf_{n\to+\infty} F(\sum_{p=0}^{n} u.\chi_{A_p}, v) \\ &\leq \liminf_{n\to+\infty} \sum_{p=0}^{n} F(u.\chi_{A_p}, v) \end{aligned}$$

It follows that

$$\begin{aligned} \liminf_{n\to+\infty} \sum_{p=0}^{n} F(u.\chi_{A_p}, v) &\geq F(u.\chi_A, v) \\ &\geq \limsup_{n\to+\infty} \sum_{p=0}^{n} F(u.\chi_{A_p}, v) + \liminf_{n\to+\infty} F(u.\chi_{B_n}, v) \\ &\geq \limsup_{n\to+\infty} \sum_{p=0}^{n} F(u.\chi_{A_p}, v) \end{aligned}$$

That is

$$F(u.\chi_A, v) = \lim_{n\to+\infty} \sum_{p=0}^{n} F(u.\chi_{A_p}, v) \quad \blacksquare$$

Lemma 1.4. *Let X be a separable Banach space and $\varphi : X \longrightarrow \overline{I\!R}$ a non negative function. The following assertions are equivalent:*

i) *φ is strongly coercive;*
ii) *φ^{**} is strongly coercive.*

Proof. The implication *ii)* $\Rightarrow$ *i)* is obvious because $\varphi^{**} \leq \varphi$. Suppose that φ is strongly coercive, then:

$$\forall \alpha > 0, \quad \exists B_\alpha > 0, \quad \text{such that } \|y\| \geq B_\alpha \Rightarrow \varphi(y) \geq \alpha\|y\|$$

and we have, $\sup_{\|y\|\geq B_\alpha} (\alpha\|y\| - \varphi(y)) \leq 0$. From another hand, φ is non negative then, $\sup_{\|y\|\leq B_\alpha} (\alpha\|y\| - \varphi(y)) \leq \alpha B_\alpha$. It follows that for every $y \in Y$, $\varphi(y) \geq \alpha\|y\| - \alpha B_\alpha$. Taking the bipolar we will have, $\varphi^{**}(y) \geq \alpha\|y\| - \alpha B_\alpha$. Which implies that for every $\alpha > 0$, we have $\lim_{\|y\|\to+\infty} \frac{\varphi^{**}(y)}{\|y\|} \geq \alpha$. $\blacksquare$

Lemma 1.5. *Let $(\alpha_n)_n$ be a sequence in $L^0_{\overline{I\!R}}(\Omega, \mu)$ and $\alpha \in L^1_{I\!R^+}(\Omega, \mu)$. If for μ-almost every ω in Ω we have $\liminf_{n\to+\infty} \alpha_n(\omega) \geq -\alpha(\omega)$. Then,*

$$\limsup_{n\to+\infty} \int_\Omega \alpha_n^-(\omega) d\mu > -\infty.$$

where $\alpha_n^- = min(\alpha_n, 0)$.

Proof. Let $a \in L^1{}_{\mathbb{R}_+^*}(\Omega, \mu)$. We have $\liminf\limits_{n\to+\infty} \alpha_n(\omega) \geq -\alpha(\omega) - a(\omega)$ and for every $\omega \in \Omega$, there exists $n_\omega \in \mathbb{N}$, such that $\alpha_n(\omega) > \alpha(\omega) - a(\omega)$ for every $n \geq n_\omega$. Set

$$A_p = \bigcap_{n\geq p} \{\omega \in \Omega, \quad \alpha_n(\omega) > \alpha(\omega) - a(\omega)\} \qquad \text{and note that } \Omega = \bigcup_p A_p.$$

If $(\Omega_p)_p$ is a σ-finite μ-recouvrement of Ω, then the sequence $(\Omega_p \cap A_p)_p$ is still a σ-finite μ-recouvrement of Ω and for every $n \in \mathbb{N}$ we have,

$$\int\limits_{\Omega} \alpha_n^-(\omega) d\mu = \lim_{p\to+\infty} \int\limits_{\Omega_p\cap A_p} \alpha_n^-(\omega) d\mu.$$

Using a diagonalization lemma due to Attouch[1], we get that there exists an increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that

$$\limsup_{n\to+\infty} \int\limits_{\Omega} \alpha_n^-(\omega) d\mu = \limsup_{n\to+\infty} \lim_{p\to+\infty} \int\limits_{\Omega_p\cap A_p} \alpha_n^-(\omega) d\mu \tag{3}$$

$$\geq \limsup_{p\to+\infty} \int\limits_{\Omega_p\cap A_p} \alpha_{\theta(p)}^-(\omega) d\mu. \tag{4}$$

Notice that for every $p \in \mathbb{N}$ and μ-almost every ω in $\Omega_p \cap A_p$, we have:

$$\alpha_{\theta(p)}(\omega) > \alpha(\omega) - a(\omega),$$

which implies:

$$\alpha_{\theta(p)}^-(\omega) > \alpha(\omega) - a(\omega).$$

Consequently:

$$\begin{aligned} \int\limits_{\Omega_p\cap A_p} \alpha_n^-(\omega) d\mu &> - \int\limits_{\Omega_p\cap A_p} (\alpha(\omega) + a(\omega)) d\mu \\ &> - \int\limits_{\Omega} (\alpha(\omega) + a(\omega)) d\mu. \end{aligned}$$

It follows that,

$$\limsup_{p\to+\infty} \int\limits_{\Omega_p\cap A_p} \alpha_{\theta(p)}^-(\omega) d\mu > -\infty$$

Then, we obtain the desired result in virtue of (4). $\blacksquare$

In the following result we give a more general version of lemma 3.7 of [4].

Lemma 1.6. *Let $(\mu_n)_n$ be a sequence of non negative atomless measures, such that for every $n \in I\!N$, $\mu_n(\Omega) = +\infty$. Then, there exists a subsequence $(\mu_{n_k})_k$ and a family $(A_k)_k$ of pairwise disjoints measurable subsets such that:*

$$\mu_{n_k}(A_k) = +\infty \qquad \forall k \in I\!N.$$

Proof. We adapt the proof of ([4], lemma 3.7) to non absolutely continuous measures. ∎

Lemma 1.7. *Let $\psi : \Omega \times Y \to I\!R$ be a Young function and $g : \Omega \times Y' \to I\!R$. If $v' \in domI_{\psi^*}$ and $A \in \mathcal{A}$ are such that $\int\limits_A g(v')d\mu = +\infty$. Then, for every $a > 0$, there exists a measurable subset A' in A with the following property:*

$$\int_{A'} \psi^*(\omega, v'(\omega))d\mu \leq a \quad \textit{and} \quad \int\limits_{A'} g(\omega, v'(\omega))d\mu = +\infty.$$

Proof. Suppose that the conclusion of the lemma does not hold. Then, for every measurable subset $A' \subset A$ we have one of the following alternative cases:

$$i) \int_{A'} \psi^*(\omega, v'(\omega))d\mu > a \quad ii) \int\limits_{A'} g(\omega, v'(\omega))d\mu < +\infty \tag{5}$$

First, we note that $\int_A \psi^*(\omega, v'(\omega))d\mu > a$ because $\int\limits_A g(\omega, v'(\omega))d\mu = +\infty$. Since the measure μ is atomless, there exists a measurable subset A_1 of A such that $\int_{A_1} \psi^*(\omega, v'(\omega))d\mu = a$. Using (5) we will have $\int\limits_{A_1} g(\omega, v'(\omega))d\mu < +\infty$, and consequently $\int\limits_{A\backslash A_1} g(v')d\mu = +\infty$. Furthermore we have $\int\limits_A g(\omega, v'(\omega))d\mu = \int\limits_{A\backslash A_1} g(v')d\mu + \int\limits_{A_1} g(\omega, v'(\omega)')d\mu = +\infty$, which implies, in virtue of (5) that $\int\limits_{A\backslash A_1} g(\omega, v'(\omega))d\mu = +\infty$. Using the same argument as above, we can say that there exists a measurable subset A_2 of $A\backslash A_1$ such that $\int\limits_{A_2} \psi^*(\omega, v'(\omega))d\mu = a$. Using successively this method we can build a family $(A_k)_k$ of pairwise disjoints measurable subsets such that $\int\limits_{A_k} \psi^*(\omega, v'(\omega))d\mu = a$ for every $k \in \mathbb{N}$. It follows that

$$\begin{aligned} \int\limits_{\Omega} \psi^*(\omega, v'(\omega))d\mu \ &\geq \int\limits_{\bigcup\limits_k A_k} \psi^*(\omega, v'(\omega))d\mu \\ &= {}_{k}\int\limits_{A_k} \psi^*(\omega, v'(\omega))d\mu = +\infty \end{aligned}$$

Which contradicts the fact that $v' \in domI_{\psi^*}$. ■

2. Inf-compactness of an integral functional

The aim of this section is to study the Inf-compactness property of an integral functional defined on a product of spaces. Note that for the one variable functionals there exists several characterizations. We recall the following which is due to Giner ([12], Th 5.2.6).

Theorem 2.1. [13] *Let Y be a reflexive separable Banach space, $\psi : \Omega \times Y \to \overline{\mathbb{R}}_+$ a Young function, and $f : \Omega \times Y \to \overline{\mathbb{R}}$ a normal convex integrand. Consider the following assertions:*

i) *The functional I_f is proper $\sigma(L_\psi, L_{\psi^*})$ inf-compact ;*
ii) *f^* is almost essentially dominated by ψ^* ;*
iii) *L_{ψ^*} is a subset of $domI_{f^*}$;*
iv) $\lim\limits_{\|v\|_\psi \to +\infty} \frac{I_f(v)}{\|v\|_\psi} = +\infty.$

Then, i) $\Rightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) and if I_f is not everywhere equal to $+\infty$, all assertions are equivalent.

When u is fixed in any integral space, the inf-compactness of $I_f(u, .)$ is characterized by the above theorem. All the conditions in this theorem depends on u. In the following, we give an analogous characterization uniformly in u, when u run over a decomposable set.

Theorem 2.2. *Let X and Y be two separable Banach spaces, whit the additional condition that Y is reflexive. $\Gamma : \Omega \to X$ is a measurable multiapplication, $\psi : \Omega \times Y \to \overline{\mathbb{R}}$ is a Young function and $f : \Omega \times X \times Y \to \overline{\mathbb{R}}$ is an integrand such that, for μ-almost every ω in Ω, we have:*

i) *f is $(\|.\|, \sigma)$-lsc on $X \times Y$;*
ii) *$f(\omega, x, .)$ is convex for every $x \in X$.*

Let $D \subset L^0_X(\Omega,\mu)$ *be a rich* σ*-decomposable subset of* L^0_Γ*. Let us put*

$$g(\omega,y) = \inf_{x\in\Gamma(\omega)} f(\omega,x,y)$$

If for every $u \in D$ *the functional* $I_f(u,.)$ *is not everywhere equal to* $+\infty$*, then the following assertions are equivalents:*

a) $I_f(u,.)$ *is proper* $\sigma(L_\psi, L_{\psi^*})$ *inf-compact for every* $u \in D$ *;*
b) I_g *is proper* $\sigma(L_\psi, L_{\psi^*})$ *inf-compact;*
c) *For every* $\lambda > 0$ *there exists* $b_\lambda > 0$ *and* $a_\lambda \in L^1{}_{I\!R^+}(\Omega,\mu)$ *such that*

$$f(\omega,x,y) \geq \psi(\omega,\frac{\lambda y}{b_\lambda}) + a_\lambda(\omega) \qquad \forall (x,y)\in\Gamma(\omega)\times Y.$$

Proof. *a)* $\Rightarrow$ *b)* Suppose that $I_f(u,.)$ is proper and $\sigma(L_\psi, L_{\psi^*})$ inf-compact for every $u \in D$. Using theorem (2.1), we will have

$$L_{\psi^*} \subset domI_{f_u^*} \qquad \forall u \in D. \tag{6}$$

Applying proposition (1.2) to the functional $F(u,v) = I_{f_u^*}(v)$ we deduce that

$$L_{\psi^*} \subset dom(\sup_{u\in D} I_{f_u^*}). \tag{7}$$

But $g(\omega,y) = \inf\limits_{x\in\Gamma(\omega)} f(\omega,x,y)$ then, we will have

$$\sup_{u\in D} I_{f_u^*}(v') = I_{g^*}(v') \quad \text{for every } v' \in L_{\psi^*}. \tag{8}$$

Indeed, let $v' \in L_{\psi^*}$. The inclusion (6) implies that $I_{f_u^*}(v') < +\infty$ for every $u \in L^0_\Gamma$ and for every $v \in L_\psi$ such that $I_f(u,v) \in$ IR, we can write

$$I_{f_u^*}(v') \geq \int_\Omega < v'(\omega), v(\omega) > d\mu - \int_\Omega f(\omega,u(\omega),v(\omega))d\mu > -\infty.$$

So, $f_u^*(v') \in {L^1}_{\mathbb{R}}(\Omega,\mu)$. Regarding to theorem 1.4 of [?], we will have :

$$\begin{aligned}
\sup_{u\in D} I_{f_u^*}(v') &= \sup_{u\in D}\int_\Omega \sup_{y\in Y}\left(< v'(\omega), y > - f(\omega, u(\omega), y)\right) d\mu \\
&= \int_\Omega \sup_{x\in\Gamma(\omega)}\sup_{y\in Y}\left(< v'(\omega), y > - f(\omega, x, y)\right) d\mu \\
&= \int_\Omega \sup_{y\in Y}\left(< v'(\omega), y > - \inf_{x\in\Gamma(\omega)} f(\omega, x, y)\right) d\mu \\
&= \int_\Omega g^*(\omega, v'(\omega)) d\mu \\
&= I_{g^*}(v');
\end{aligned}$$

and we conclude that I_g is $\sigma(L_\psi, L_{\psi^*})$ inf-compact by virtue of theorem (2.1).
b) $\Longrightarrow$ *a)* is obvious since $f(\omega, u(\omega), y) \geq g(\omega, y)$ for every $u \in D$ and every $y \in Y$.
b) $\Longleftrightarrow$ *c)* by virtue of theorem (2.1)*ii*).

Corollary 2.3. *Under the same hypothesis of theorem (2.2), if I_{f_u} is $\sigma(L_\psi, L_{\psi^*})$ inf-compact for every u in a rich decomposable subset D of L^0_Γ, then $\bigcup_{u\in D}\{I_{f_u} \leq \lambda\}$ is weakly relatively compact for every $\lambda \in \mathbb{R}$.*

Proof. Under the same notation as above, observe that $\bigcup_{u\in D}\{I_{f_u} \leq \lambda\} \subset \{I_g \leq \lambda\}$ and we conclude with theorem (2.2). ■

In the particular case where u runs over the $L^\infty_X(\Omega,\mu)$ space, we obtain the following corollary:

Corollary 2.4. *Let X and Y be two separable Banach spaces, with the additional condition that Y is reflexive. $\psi : \Omega\times Y \to \overline{\mathbb{R}}$ be a Young function, $f : \Omega\times X\times Y \to \overline{\mathbb{R}}$ be an integrand with the same hypothesis as in theorem (2.2). Suppose that for every $u \in L^\infty_X(\Omega,\mu)$ the functional $I_f(u, .)$ is not everywhere equal to $+\infty$. Then, the following assertions are equivalents;*

i) *For every $u \in L^\infty_X(\Omega,\mu)$, the functional $I_f(u, .)$ is proper $\sigma(L_\psi, L_{\psi^*})$ inf-compact;*
ii) *For every $M > 0$, the functional I_{f_M} is proper $\sigma(L_\psi, L_{\psi^*})$ inf-compact. Where,*

$$f_M(\omega, y) = \inf_{\|x\|\leq M} f(\omega, x, y).$$

Proof. For every $M > 0$, set

$$\Gamma_M(\omega) = \{x \in X, \quad \|x\| \leq M\}.$$

Then, $L^0_{\Gamma_M} = B_\infty(0, M)$ where $B_\infty(0, M)$ is the ball centered in 0 with radius M in $L^\infty_X(\Omega, \mu)$. It is sufficient to note that $L^\infty_X(\Omega, \mu) = \bigcup\limits_{M>0} B_\infty(0, M)$ and to use theorem (2.2). ■

The assertion $c)$ in theorem (2.2) implies that the function $g(\omega, y) = \inf\limits_{x\in\Gamma(\omega)} f(\omega, x, y)$ is strongly coercive for μ-almost every ω in Ω. The reciprocal result seems to be true in many situations. More precisely, we have the next proposition. First, Recall the following technical lemmas which are due to Giner ([12]).

Lemma 2.5. *([12], 4.1.3) Let φ be a Young function which is continuous in the origin and strongly coercive. Then, there exists another Young function ψ which satisfies a Δ_2-condition, such that ψ is almost essentially dominated by φ. Moreover, the following topological inclusion hold*

$$(L_\varphi, \sigma(L_\varphi, E_{\varphi^*})) \hookrightarrow (L_\psi, \sigma(L_\psi, L_{\psi^*})) = (L_\psi, \sigma(L_\psi, L'_\psi)). \tag{9}$$

Lemma 2.6. *([12], 4.2.4) Let φ be a Young function. Then, Bounded closed convex subsets of $L_\varphi(\Omega, \mu)$ are exactly those subsets which are $\sigma(L_\varphi, E_{\varphi^*})$-compacts.*

Proposition 2.7. *Let X and Y be two separable Banach spaces, with the additional condition that Y is reflexive. $\Gamma : \Omega \to X$ a measurable multi-application and $f : \Omega \times X \times Y \to \overline{I\!R}$ an integrand with the same hypothesis as in theorem (2.2). Suppose that the function $g : \Omega \times Y \to \overline{I\!R}$ where $g(\omega, y) = \inf\limits_{x\in\Gamma(\omega)} f(\omega, x, y)$ is strongly coercive for μ-almost every ω in Ω and that $g^*(., 0) \in {L^1}_{I\!R}(\Omega, \mu)$. Then, there exists $\psi : \Omega \times Y \to \overline{I\!R}$ a Young function which satisfies a Δ_2-condition such that for every $u \in L^0_\Gamma$ the functional $I_f(u, .)$ is $\sigma(L_\psi, L_{\psi^*})$ inf-compact.*

Proof. Set $h(\omega, .) = g(\omega, .) + g^*(\omega, 0)$, then h is a non negative strongly coercive function, by virtue of lemma (1.4), $h^{**}(\omega, .)$ is also strongly coercive for μ-almost every ω in Ω. Since $g^*(\omega, 0)$ is finite almost everywhere, $g^{**}(\omega, .) = h^{**}(\omega, .) - g^*(\omega, .)$ is also strongly coercive. Using theorem 3.3 of [11], we get that there exists a strongly coercive Young function φ such that

$$g^{**}(\omega, y) \geq \varphi(\omega, y) - g^*(\omega, 0) \qquad \forall(\omega, y) \in \Omega \times Y.$$

For every $u \in L^0_\Gamma$ and every $v \in L_\psi$ we have, by Using (2.5), that

$$\begin{aligned} f(\omega, u(\omega), v(\omega)) &\geq g(\omega, v(\omega)) \\ &\geq g^{**}(\omega, v(\omega)) \\ &\geq \varphi(\omega, v(\omega)) - g^*(\omega, 0) \end{aligned}$$

for μ-almost ω in Ω. Then,

$$I_f(u,v) \geq I_\varphi(v) - \int\limits_\Omega g^*(\omega,0)d\mu$$

which shows that lower sections of $I_f(u,.)$ are bounded in L_φ. Furthermore, the function $f(u(.),.)$ is lsc and bounded below by $-g^*(.,0)$ which is integrable. Thus, the lower sections of $I_f(u,.)$ are also closed in measure and convex in L_φ. By using (2.6), we conclude that such sections are $\sigma(L_\varphi, E_{\varphi^*})$-compacts. In virtue of (9) it follows that they are $\sigma(L_\psi, L'_\psi)$-compacts. Which completes the proof. ■

Theorem 2.2 gives inf-compactness characterization of integral functional I_{f_u} for every u in a decomposable set. Now, we give a new inf-compactness condition of I_{f_u} with u in a neighborhood of u_o. In the following definition we will introduce a new notion of equi-lower semicontinuity in the neighborhood of $+\infty$. This last notion will play an important role in our study.

Definition 2.8. *Let X be a topological space and Y be a normed vector space. We say that a function $f : X \times Y \to \overline{I\!R}$ is equi-lsc in the neighborhood of $+\infty$ at $x_0 \in X$ if:*

$\forall \epsilon > 0 \ \ \exists V_\epsilon \in \mathcal{V}(x_0) \ \ \exists r_\epsilon > 0$ *such that* $f(x_0, y) \leq f(x,y) + \epsilon \, \forall (x,y) \in V_\epsilon \times Y, \|y\| > r_\epsilon$;

where $\mathcal{V}(x_0)$ denotes the set of all neighborhoods of x_0.

Example. Let $f(\omega,x,y) = p(\omega,x) + q(w,y)$ where $p : \Omega \times X \longrightarrow \overline{\mathrm{I\!R}}$ and $q : \Omega \times Y \longrightarrow \overline{\mathrm{I\!R}}$. If p is lower semicontinuous at x_0 then f is equi-lsc in the neighborhood of $+\infty$ at x_0.

Under this assumption of equi-lsc in the neighborhood of $+\infty$, the strong coercivity of a function in a point implies its "uniform" strong coercivity in a neighborhood of this point:

Lemma 2.9. *Let X be a hausdorff topological space and Y be a normed space $f : X \times Y \to \overline{I\!R}$ and $x_0 \in X$. Suppose that:*

i) *f is equi-lsc in the neighborhood of $+\infty$ at x_0.*
ii) *$f(x_0,.)$ is strongly coercive.*

Then, there exists a neighborhood V of x_0 in X such that $\inf\limits_{x \in V} f(x,.)$ is strongly coercive.

Proof. Let $\epsilon > 0$. According to i), there exists $V \in \mathcal{V}(x_o)$ and $r_\epsilon > 0$ such that $\inf\limits_{x \in V} f(x,y) > f(x_0,y) - \epsilon$ for every $\|y\| > r_\epsilon$. Since $f(x_0,.) - \epsilon$ is strongly coercive, it will be the same for $\inf\limits_{x\in V} f(x,.)$. ■

Generally, the upper bound of a family of upper semicontinuous functions (usc) is not necessary usc, but with the assumption of equi-lsc in the neighborhood of $+\infty$ we get back the scs of the following function: $x \in X \to f_x^*(y') = \sup\limits_{y\in Y}(< y, y' > -f(x,y)) \in \overline{\mathbb{R}}$ for every fixed $y' \in Y$. Indeed, we have the following result:

Proposition 2.10. *Let X and Y be two separable Banach spaces, with the additional hypothesis that Y is reflexive, $f : X \times Y \to \overline{\mathbb{R}}$ a function which satisfy:*

i) *f is $(\|.\|, \sigma)$-lsc on $X \times Y$;*
ii) *f is equi-lsc in the neighborhood of $+\infty$ at x_0.*

Then, for every $y' \in Y'$, the function $x \in X \to f_x^*(y') \in \overline{\mathbb{R}}$ is norm usc at x_0.

Proof. Let $y' \in Y'$ be fixed. We must show that the function:

$$h(x) = -f_x^*(y') = \inf_{y\in Y}(f(x,y) - < y, y' >)$$

is lsc at x_0. Let $\epsilon > 0$ and $(x_n)_n \longrightarrow x_0$ in the norm of X. The hypothesis of equi-lsc in the neighborhood of $+\infty$ implies that there exists $n_1 \in \mathbb{N}$ and $B_\epsilon > 0$ such that for every $n \geq n_1$ and every $\|y\| > B_\epsilon$, we have $f(x_n, y) \geq f(x_0, y) - \epsilon$. Then

$$f(x_n, y) - < y', y > \geq f(x_0, y) - < y', y > -\epsilon \geq h(x_0) - \epsilon$$

for every $\|y\| > B_\epsilon$. It follows that,

$$\inf_{\|y\|>B_\epsilon}(f(x_n, y) - < y', y >) \geq h(x_0) - \epsilon \qquad \forall n \geq n_1. \tag{10}$$

From another hand, for every $n \in \mathbb{N}$, the function $y \in Y \to f(x_n, y) - < y', y > \in \overline{\mathbb{R}}$ reach its minimum on the ball $\{y \in Y, \quad \|y\| \leq B_\epsilon\}$ since this function is weakly lsc and Y is reflexive. Therefore, for every $n \in \mathbb{N}$, there exists $y_n \in Y$, $\|y_n\| \leq B_\varepsilon$ such that

$$\inf_{\|y\|\leq B_\epsilon}(f(x_n, y) - < y', y >) = f(x_n, y_n) - < y', y_n > \tag{11}$$

So, we will have:

$$h(x_0) \leq \liminf_{n\to+\infty}(f(x_n, y_n) - < y', y_n >) \tag{12}$$

Indeed, if the second limb is $+\infty$ there is nothing to check. Otherwise, there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that :

$$\liminf_{n\to+\infty}(f(x_n, y_n) - < y', y_n >) = \lim_{n\to+\infty}\left(f(x_{\varphi(n)}, y_{\varphi(n)}) - < y', y_{\varphi(n)} >\right) = c < +\infty.$$

The sequence $(y_{\varphi(n)})_n$ is bounded in Y, we can extract a weakly convergent subsequence $(y_{\psi(\varphi(n))})_n$ which converges to some y_0 in Y. Since the function $f(.,.)- < y',. >$ is $(\|.\|,\sigma)$-lsc on $X \times Y$ then,

$$\begin{aligned} h(x_0) &\leq f(x_0,y_0)- < y',y_0 > \\ &\leq \liminf_{n\to+\infty} \big(f(x_{\psi(\varphi(n))},y_{\psi(\varphi(n))})- < y',y_{\psi(\varphi(n))} >\big) = c \end{aligned}$$

thus (12) is satisfied and there exists $n_2 \in \mathbb{N}$ such that:

$$f(x_n,y_n)- < y',y_n > \geq h(x_0) - \epsilon \qquad \text{pour } n \geq n_2 \tag{13}$$

by virtue of (10), (11) and (13) there exists an integer $n_0 = max(n_1,n_2)$ such that:

$$h(x_n) = \inf_{y\in Y} (f(x_n,y)- < y',y >) \geq h(x_0) - \epsilon \qquad \forall n \geq n_0.$$

Which completes the proof of the upper semicontinuity of h and consequently $f_x^*(y') = -h(x)$ is lsc at x_0. ■

In the following, we check out the fact that if f is equi-lsc in neighborhood of $+\infty$ and I_f is inf-compact in a point, then the functional $I_f(u,.)$ is inf-compact for every u in a neighborhood of this point.

Theorem 2.11. *Let X and Y be two separable Banach spaces with the additional hypothesis that Y is reflexive. Let $\varphi : \Omega \times X \to \overline{\mathbb{R}}_+$ and $\psi : \Omega \times Y \to \overline{\mathbb{R}}_+$ two Young functions such that ψ^* satisfies a Δ_2-condition, and $f : \Omega \times X \times Y \to \overline{\mathbb{R}}$ is an integrand. Suppose that*

- i) *$f(\omega,.,.)$ is $(\|.\|,\sigma)$-lsc on $X \times Y$ for μ-almost every ω in Ω;*
- ii) *The function $f(\omega,x,.)$ is convex for every $x \in X$ for μ-almost every ω in Ω;*
- iii) *There exists u_0 in L_φ such that, for μ-almost every ω in Ω, the function $f(\omega,.,.)$ is equi-lsc in neighborhood of $+\infty$ at $u_0(\omega)$.*
- iv) *The functional I_f takes no $-\infty$ values.*
- v) *$I_{f_{u_0}}$ is proper $\sigma(L_\psi,L_{\psi^*})$ inf-compact.*

Then, there exists a neighborhood V_0 of u_0 in L_φ such that $I_f(u,.)$ is proper $\sigma(L_\psi,L_{\psi^*})$ inf-compact for every $u \in V_0$ such that $I_f(u,.)$ is not equal to $+\infty$.

Proof. In virtue of theorem (2.1), it is sufficient to check that:

(14)

$$\exists r > 0, \quad \text{such that} \quad L_{\psi^*} \subset domI_{f_u^*} \quad \text{for every } u \in B_\varphi(u_0,r) \quad \text{such that } domI_f(u,.) \neq \emptyset$$

where $B_\varphi(u_0,r)$ denotes the ball of L_φ centered in u_0 with radius r. Since the functional $I_f(u_0,.)$ is proper $\sigma(L_\psi,L_{\psi^*})$ inf-compact, Using theorem (2.1), we get

(15)

$$\forall\lambda > 0 \ \exists b_\lambda > 0 \ \exists a_\lambda \in {L^1}_{\mathbb{R}^+}(\Omega,\mu) \quad \text{such that} \quad f_{u_0}^*(\omega,\lambda y') \leq b_\lambda \psi^*(\omega,y') + a(\omega).$$

For every $y' \in Y'$ and μ-almost every ω in Ω.
Which is equivalent for every $y \in Y$ and μ-almost every ω in Ω to

(16)

$$\forall \lambda > 0 \ \exists b_\lambda > 0 \ \exists a_\lambda \in L^1_{\mathbb{R}^+}(\Omega, \mu) \quad \text{such that} \quad f(\omega, u_0(\omega), y) \geq b_\lambda \psi(\omega, \frac{\lambda y}{b_\lambda}) - a_\lambda(\omega).$$

Suppose that (14) does not hold. Then, there would exist two sequences $(u_n)_n$ in L_φ and $(v'_n)_n$ in L_{ψ^*} such that:

$$\|u_n - u_0\|_\varphi \leq \frac{1}{n}, \quad I_{f^*_{u_n}}(v'_n) = +\infty \quad \text{and} \quad I_f(u_n, .) \neq +\infty. \tag{17}$$

For every $n \in \mathbb{N}^*$. Set

$$\begin{aligned} B_n &= \{\omega \in \Omega \quad f^*_{u_n}(\omega, v'_n(\omega)) \leq 0\} && (18)\\ \bar{u}_n &= u_n.\chi_{B_n} + u_0.\chi_{B_n^c} && (19)\\ \bar{v}'_n &= v'_n.\chi_{B_n} && (20) \end{aligned}$$

The function $f^*_{\bar{u}_n}(\omega, \bar{v}'_n(\omega)) = f^*_{u_n}(\omega, v'_n(\omega)).\chi_{B_n} + f^*_{u_0}(\omega, 0).\chi_{B_n^c}$ is well defined, because if $\omega \in \Omega \backslash B_n$ then:

$$\begin{aligned} f^*_{\bar{u}_n}(\omega, \bar{v}'_n(\omega)) &= f^*_{u_0}(\omega, 0)\\ &= \sup_{y \in Y} (-f(\omega, u_0(\omega), y))\\ &\geq -f(\omega, u_0(\omega), v_0(\omega)) > -\infty \end{aligned}$$

Where v_0 is some element of L_ψ such that $I_f(u_0, v_0) \in \mathbb{R}$.
From another hand, we have in virtue of (16),

$$f^*_{\bar{u}_n}(\omega, \bar{v}'_n(\omega)) = \sup_{y \in Y} (-f(\omega, u_0(\omega), y)) \leq a_\lambda(\omega) < +\infty$$

Thus, for every n in $\mathbb{N}$, we have $f^*_{\bar{u}_n}(\omega, \bar{v}'_n(\omega)).\chi_{B_n} \in L^0_{\overline{\mathbb{R}}_+}(\Omega, \mu)$ and $f^*_{u_n}(., v'_n(.)) \geq 0$ on B_n^c. Then we get

$$\begin{aligned} \int_\Omega f^*_{\bar{u}_n}(\omega, \bar{v}'_n(\omega)).\chi_{B_n} d\mu &= \int_{B_n} f^*_{u_n}(\omega, v'_n(\omega)) d\mu\\ &= \int_\Omega f^*_{u_n}(\omega, v'_n(\omega)) d\mu - \int_{B_n^c} f^*_{u_n}(\omega, v'_n(\omega)) d\mu\\ &= I_{f^*_{u_n}}(v'_n) - \int_{B_n^c} f^*_{u_n}(\omega, v'_n(\omega)) d\mu\\ &= +\infty. \end{aligned}$$

Applying lemma (1.6) to the non-negative measures $\mu_n(.) = \int\limits_{(.)} f^*_{\bar{u}_n}(\omega, \bar{v}'_n(\omega)).\chi_{B_n} d\mu$, we can extract a subsequence $(\bar{u}_{n_k})_k$ and build a family $(A'_k)_k$ of pairwise disjoint measurable subsets such that

$$\int\limits_{A'_k} f^*_{\bar{u}_{n_k}}(\omega, \bar{v}'_{n_k}(\omega)).\chi_{B_{n_k}} d\mu = +\infty;$$

for every $k \in \mathbb{N}$. Set $A_k = A'_k \cap B_{n_k}$, then $(A_k)_{k\in\mathbb{N}}$ is a family of pairwise disjoint measurable subsets such that

$$\int\limits_{A_k} f^*_{u_{n_k}}(\omega, v'_{n_k}(\omega)) d\mu = +\infty \qquad \text{for every } k \in \mathbb{N}$$

but $v'_{n_k} \in L_{\psi^*} = domI_{\psi^*}$ then, for every $k \in \mathbb{N}^*$ lemma (1.7) insure the existence of a measurable subset $E_k \subset A_k$ such that:

$$\int\limits_{E_k} \psi^*(\omega, v'_{n_k}(\omega)) d\mu \leq \frac{1}{2^k} \quad \text{and} \quad \int\limits_{E_k} f^*_{u_{n_k}}(\omega, v'_{n_k}(\omega)) d\mu = +\infty. \tag{21}$$

Let us define the following functions:

$$v' = \sum_n v'_{n_k}.\chi_{E_k} \quad \text{and} \quad w_p = u_{n_p}.\chi_{\bigcup\limits_k E_k} + u_o.\chi_{\Omega\setminus\bigcup\limits_k E_k}. \tag{22}$$

Notice that $v' \in L_{\psi^*}$ because

$$\int_\Omega \psi^*(\omega, v'(\omega)) d\mu = \sum_k \int\limits_{E_k} \psi^*(\omega, v'(\omega)) d\mu \leq \sum_k \frac{1}{2^k} < +\infty$$

and for every $p \in \mathbb{N}$ we have

$$\begin{aligned}
\int\limits_{\bigcup\limits_{k\neq p} E_k} f^*_{u_{n_p}}(\omega, v'(\omega)) d\mu &= \int\limits_{\bigcup\limits_{k\neq p} E_k} \sup_{y\in Y} \big(< v'(\omega), y > - f(\omega, u_{n_p}(\omega), y)\big) d\mu \\
&\geq \int\limits_{\bigcup\limits_{k\neq p} E_k} \big(< v'(\omega), v_{n_p}(\omega) > - f(\omega, u_{n_p}(\omega), v_{n_p}(\omega))\big) d\mu \\
&\geq -\int\limits_{\Omega} | < v'(\omega), v_{n_p} > | d\mu - \int\limits_{\Omega} |f(\omega, u_{n_p}(\omega), v_{n_p}(\omega))| d\mu \\
&> -\infty. \qquad (23)
\end{aligned}$$

Where $v_{n_p} \in L_\psi$ is such that $I_f(u_{n_p}, v_{n_p}) \in \mathbb{R}$. This choice is possible because $I_f(.,.) > -\infty$ and $I_f(u_{n_p}, .)$ is not everywhere equal to $+\infty$.
For every $p \in \mathbb{N}$, In virtue of (21) and (23) we can write:

$$\begin{aligned}
\int_\Omega f^*_{w_p}(\omega, v'(\omega))d\mu &= \int_{\bigcup_k E_k} f^*_{u_{n_p}}(\omega, v'(\omega))d\mu + \int_{\Omega\setminus\bigcup_k E_k} f^*_{u_0}(\omega, v'(\omega))d\mu \\
&\geq \int_{E_p} f^*_{u_{n_p}}(\omega, v'(\omega))d\mu + \int_{\bigcup_{k\neq p} E_k} f^*_{u_{n_p}}(\omega, v'(\omega))d\mu + \int_\Omega |f(\omega, u_0(\omega), v_0(\omega))|d\mu \\
&= +\infty.
\end{aligned}$$

So,

$$\int_\Omega f^*_{w_p}(\omega, v'(\omega))d\mu = +\infty \qquad \text{for every } p \in \mathbb{N} \tag{24}$$

From another hand, using (17) and (22), $(w_p)_p$ converges to u_0 in L_φ and, if necessary take a subsequence, we can suppose that $(w_p)_p$ converges μ-almost everywhere to u_0. Since for μ-almost every ω in Ω, the function $x \to f^*_x(\omega, v'(\omega))$ is usc in $u_0(\omega)$ (proposition (2.10)) then,

$$\limsup_{p\to+\infty} f^*_{w_p(\omega)}(\omega, v'(\omega)) \leq f^*_{u_0(\omega)}(\omega, v'(\omega)) \qquad \mu\text{-almost every where}$$

Taking $\lambda = 1$ in (15), there exists $b_1 > 0$ and $a_1 \in L^1{}_{\mathbb{R}_+}(\Omega, \mu)$ such that:

$$\limsup_{p\to+\infty} f^*_{w_p(\omega)}(\omega, v'(\omega)) \leq f^*_{u_0(\omega)}(\omega, v'(\omega)) \leq b_1\psi^*(\omega, v'(\omega)) + a_1(\omega)$$

that is

$$-(b_1\psi^*(\omega, v'(\omega)) + a_1(\omega)) \leq \liminf_{p\to+\infty}(-f^*_{w_p(\omega)}(\omega, v'(\omega)))$$

Finally, Notice that $-(b_1\psi^*(\omega, v'(\omega)) + a_1(\omega)) \in L^1{}_{\mathbb{R}_+}(\Omega, \mu)$ and in virtue of lemma (1.5), we obtain:

$$-\infty < \limsup_{p\to+\infty} \int_\Omega -f^*_{w_p}(\omega, v'(\omega))d\mu = -\liminf_{p\to+\infty} \int_\Omega f^*_{w_p}(\omega, v'(\omega))d\mu.$$

Which contradicts (24) and completes the proof. ■

Observe that the balls $B_\infty(u_0, r) = \{u \in L^\infty_X, \ \| u - u_0 \| \leq r\}$ of L^∞_X are decomposable. Combining theorems (2.2) and (2.11) gives the following corollary

Corollary 2.12. *Under the same hypothesis of theorem (2.11), if there exists* $u_0 \in L_X^\infty$ *such that* $I_{f_{u_0}}$ *is* $\sigma(L_\psi, L_{\psi^*})$*-inf-compact, then there exists* $M > 0$ *such that* I_{f_M} *is inf-compact, where* $f_M(\omega, y) = \inf\limits_{\|x-u_0(\omega)\|\leq M} f(\omega, x, y)$.

Using the same arguments of corollary (2.3), we deduce that

Corollary 2.13. *Under the same hypothesis of theorem (2.11), if* $I_{f_{u_0}}$ *is* $\sigma(L_\psi, L_{\psi^*})$*-inf-compact for some* $u_0 \in L_X^\infty$*, then there exists* $M > 0$ *such that* $\bigcup\limits_{\|u-u_0\|\leq M} \{I_{f_u} \leq \lambda\}$ *is weakly relatively compact.*

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